The moduli of representations of degree 2

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Abstract. There are 6 types of 2-dimensional representations in general. For any groups and any monoids, we can construct the moduli of 2-dimensional representations for each type: the moduli of absolutely irreducible representations, representations with Borel mold, representations with semi-simple mold, representations with unipotent mold, representations with unipotent mold over $\mathbb{F}_2$, and representations with scalar mold. We can also construct them for any associative algebras.

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1. Introduction

In this paper we deal with the moduli of representations of degree 2. We can classify 2-dimensional representations into 6 types in general. For any groups and any monoids, we can construct the moduli of 2-dimensional representations for each type. For any associative algebras, we can also construct them for each type.

In [12] we have introduced the notion of mold. A mold is, so to say, a subalgebra of the full matrix ring. More precisely, a subsheaf of $\mathcal{O}_X$-algebras $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ on a scheme X is called a *mold* if $\mathcal{A}$ is a subbundle of $\mathrm{M}_n(\mathcal{O}_X)$. Let Γ be a group or a monoid. By a homomorphism $\rho : \Gamma \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$, we understand an *$n$-dimensional representation of Γ on a scheme X*. We say that a representation ρ *has a mold* $\mathcal{A}$ if the subsheaf of $\mathcal{O}_X$-algebras $\mathcal{O}_X[\rho(\Gamma)]$ of $\mathrm{M}_n(\mathcal{O}_X)$ generated by $\rho(\Gamma)$ coincides with $\mathcal{A}$. It is effective to classify representations with respect to molds for constructing the moduli of equivalence classes of representations. If we try to construct the moduli of equivalence classes of all representations without classifying representations with respect to molds, then two representations which have the same composition factors coincide as points of the moduli even if they are not equivalent. For separating such representations in the moduli, we need to collect only representations which have the same mold. For

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example, we have constructed the moduli of equivalence classes of absolutely irreducible representations denoted by $\mathrm{Ch}_n(\Gamma)_{\mathrm{air}}$ in [10], where $\rho : \Gamma \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$ is *absolutely irreducible* if $\mathcal{O}_X[\rho(\Gamma)]$ coincides with the full matrix ring $\mathrm{M}_n(\mathcal{O}_X)$. We also have constructed the moduli of equivalence classes of representations with Borel mold denoted by $\mathrm{Ch}_n(\Gamma)_B$ in [12], where $\rho : \Gamma \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$ is a *representation with Borel mold* if for each $x \in X$ there exists $P \in \mathrm{GL}_n(\mathcal{O}_X(U))$ on a neighbourhood U of x such that $P \cdot \mathcal{O}_U[\rho(\Gamma)] \cdot P^{-1}$ coincides with the subsheaf of $\mathcal{O}_U$-algebras of $\mathrm{M}_n(\mathcal{O}_U)$ consisting of upper triangular matrices. The author calls the plan to construct the moduli of equivalence classes of representations for any suitable molds "mold program". In this article, we will complete the mold program of degree 2.

For k-subalgebras A and B of the full matrix ring $\mathrm{M}_2(k)$ over an algebraically closed field k, we say that A and B are *equivalent* if there exists $P \in \mathrm{GL}_2(k)$ such that $P^{-1}AP = B$. There are 5 equivalence classes of k-subalgebras A of $\mathrm{M}_2(k)$: (1) $A = \mathrm{M}_2(k)$, (2) $A = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$, (3) $A = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$, (4) $A = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in k \right\}$, (5) $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \,\middle|\, a \in k \right\}$. Let $\rho : \Gamma \to \mathrm{M}_2(k)$ be a 2-dimensional representations of a group or a monoid Γ. By equivalence classes of the subalgebra $k[\rho(\Gamma)]$ of $\mathrm{M}_2(k)$, we classify 2-dimensional representations into 6-types (not 5-types!). For each cases (1)–(5), we say that ρ is (1) an *absolutely irreducible representation*, (2) a *representation with Borel mold*, (3) a *representation with semi-simple mold*, (4) a *representation with unipotent mold*, (5) a *representation with scalar mold*, respectively. In the case (4), we need to divide representations with unipotent mold into 2 types: (4-a) when $\mathrm{ch} k \neq 2$, we say ρ is a *representation with unipotent mold*, and (4-b) when $\mathrm{ch} k = 2$, we say ρ is a *representation with unipotent mold over* $\mathbb{F}_2$. It is natural to divide the case (4) into 2 types for constructing the"good" moduli of representations with unipotent mold. Here, by constructing the "good" moduli of representations, we understand constructing smooth moduli schemes of representations at least for free monoids (more precisely, see the beginning of §5). Hence there are 6 types of 2-dimensional representations in general.

In §3, we introduce the notions of (1), (2), (3), (4-a), (4-b), (5) on 2-dimensional representations on arbitrary schemes X (Definitions 2.7, 3.4, 3.5, 3.6, and 3.8). For 2-dimensional representations ρ_1, ρ_2 on X, we say that ρ_1, ρ_2 are *equivalent* (or $\rho_1 \sim \rho_2$) if there exists a $\Gamma(X, \mathcal{O}_X)$-algebra isomorphism $\sigma : \mathrm{M}_2(\Gamma(X, \mathcal{O}_X)) \to \mathrm{M}_2(\Gamma(X, \mathcal{O}_X))$ such that $\sigma(\rho_1(\gamma)) = \rho_2(\gamma)$ for any $\gamma \in \Gamma$. If $\rho_1 \sim \rho_2$, then for each $x \in X$ there exists $P \in \mathrm{GL}_2(\mathcal{O}_X(U))$ on a neighbourhood U of x such that $P^{-1}\rho_1(\gamma)P = \rho_2(\gamma)$ on U for any $\gamma \in \Gamma$. We have constructed the moduli of equivalence classes of representations in the cases (1) absolutely irreducible representations and (2) representations with Borel mold in [10] and [12], respectively. In the case (5) representations with scalar mold, we can easily construct the moduli (Theorem 3.12). In the cases (3) representations with semi-simple mold, (4-a) representations with unipotent mold, and (4-b) representations with unipotent mold over $\mathbb{F}_2$, we have the following theorems:

Theorem 1.1 (Theorem 4.29)**.** *There exists a fine moduli scheme* $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ *associated to the sheafification* $\mathcal{E}q\mathcal{SS}_2(\Gamma)$ *of the functor*

$$\begin{array}{ccc} (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \left\{ \begin{array}{l} \text{2-}\textit{dimensional representations} \\ \textit{with semi-simple mold of } \Gamma \textit{ on } X \end{array} \right\} \Big/ \sim \end{array}$$

with respect to Zariski topology for arbitrary group or monoid Γ*. The moduli* $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ *is separated over* $\mathbb{Z}$*; if* Γ *is a finitely generated group or monoid, then* $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ *is of finite type over* $\mathbb{Z}$*.*

Theorem 1.2 (Theorem 5.23)**.** *There exists a fine moduli scheme* $\mathrm{Ch}_2(\Gamma)_u$ *associated to the sheafification* $\mathcal{E}q\,\mathcal{U}_2(\Gamma)$ *of the functor*

$$\begin{array}{ccc} (\mathbf{Sch}/\mathbb{Z}[1/2])^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \left\{ \begin{array}{l} \text{2-}\textit{dimensional representations} \\ \textit{with unipotent mold of } \Gamma \textit{ on } X \end{array} \right\} \Big/ \sim \end{array}$$

with respect to Zariski topology for arbitrary group or monoid Γ*. The moduli* $\mathrm{Ch}_2(\Gamma)_u$ *is separated over* $\mathbb{Z}[1/2]$*; if* Γ *is a finitely generated group or monoid, then* $\mathrm{Ch}_2(\Gamma)_u$ *is of finite type over* $\mathbb{Z}[1/2]$*.*

Theorem 1.3 (Theorem 6.20)**.** *There exists a fine moduli scheme* $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *associated to the sheafification* $\mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$ *of the functor*

$$\begin{array}{ccc} (\mathbf{Sch}/\mathbb{F}_2)^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \left\{ \begin{array}{l} \text{2-}\textit{dimensional representations with} \\ \textit{unipotent mold over } \mathbb{F}_2 \textit{ of } \Gamma \textit{ on } X \end{array} \right\} \Big/ \sim \end{array}$$

with respect to Zariski topology for arbitrary group or monoid Γ*. The moduli* $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *is separated over* $\mathbb{F}_2$*; if* Γ *is a finitely generated group or monoid, then* $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *is of finite type over* $\mathbb{F}_2$*.*

For any associative algebra A over any commutative ring R, we also obtain the same theorems on 2-dimensional representations of A over R:

There exist fine moduli schemes $\mathrm{Ch}_2(A)_{\text{s.s.}}$, $\mathrm{Ch}_2(A)_u$, and $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ separated over R. If A is finitely generated associative algebra over R, then the fine moduli schemes $\mathrm{Ch}_2(A)_{\text{s.s.}}$, $\mathrm{Ch}_2(A)_u$, and $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ are of finite type over R (Remarks 4.30, 5.24, 5.25, 6.22, and 6.23). These theorems are main results of this article.

As a continuation of this article, we can deal with the absolutely irreducible representations parts of the representation variety and the character variety: $\mathrm{Rep}_2(\Gamma)_{\text{air}}$ and $\mathrm{Ch}_2(\Gamma)_{\text{air}}$ in [10]. For a group or a monoid Γ, the representation variety $\mathrm{Rep}_2(\Gamma)$ is the affine scheme representing the contravariant functor which maps each scheme X to the set of 2-dimensional representations of Γ on X. For $* =$ air, B, s.s., u, $u/\mathbb{F}_2$, or scalar, $\mathrm{Rep}_2(\Gamma)_*$ denotes the subscheme of $\mathrm{Rep}_2(\Gamma)$ consisting of 2-dimensional representations with the mold corresponding to $*$. For a field k, the set of k-rational points of the representation variety $\mathrm{Rep}_2(\Gamma)$ is the disjoint union of the sets of k-rational points of $\mathrm{Rep}_2(\Gamma)_{\text{air}}$, $\mathrm{Rep}_2(\Gamma)_B$, $\mathrm{Rep}_2(\Gamma)_{\text{s.s.}}$, $\mathrm{Rep}_2(\Gamma)_u$ (or $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$), and $\mathrm{Rep}_2(\Gamma)_{\text{scalar}}$. Hence for a finitely generated group or monoid Γ and for the finite field $\mathbb{F}_q$, the number of $\mathbb{F}_q$-rational points of $\mathrm{Rep}_2(\Gamma)_{\text{air}}$ can be calculated from those of $\mathrm{Rep}_2(\Gamma)$ and the others $\mathrm{Rep}_2(\Gamma)_*$. Since $\mathrm{Rep}_2(\Gamma)_{\text{air}} \to \mathrm{Ch}_2(\Gamma)_{\text{air}}$ is a PGL_2-principal fibre bundle, the number of $\mathbb{F}_q$-rational points of $\mathrm{Ch}_2(\Gamma)_{\text{air}}$ can be also calculated from the result of $\mathrm{Rep}_2(\Gamma)_{\text{air}}$. Similarly, the virtual Hodge polynomials of $\mathrm{Rep}_2(\Gamma)_{\text{air}}$ and $\mathrm{Ch}_2(\Gamma)_{\text{air}}$ over $\mathbb{C}$ can be calculated from those of $\mathrm{Rep}_2(\Gamma)$ and the others $\mathrm{Rep}_2(\Gamma)_*$ over $\mathbb{C}$. The existence of such geometric objects as the moduli of representations with several molds helps us to understand relations between the numbers of equivalence classes of representations of Γ over $\mathbb{F}_q$ and virtual Hodge polynomials of the moduli (*cf.* [13]).

In [13], the authors deal with the case that Γ is the free monoid Υ_m of rank m. Since $\mathrm{Rep}_2(\Upsilon_m)$ is isomorphic to $\mathrm{M}_2 \times \cdots \times \mathrm{M}_2$ (m times), the numbers of the $\mathbb{F}_q$-rational points of $\mathrm{Rep}_2(\Upsilon_m)_{\text{air}}$ and $\mathrm{Ch}_2(\Upsilon_m)_{\text{air}}$ have been calculated explicitly. (The author needs to mention that our strategy to calculate the numbers of the $\mathbb{F}_q$-rational points is essentially same as [1] and [7]. Moreover, the method of [14] is much easier than our strategy.) We have also calculated the virtual Hodge polynomials of $\mathrm{Rep}_2(\Upsilon_m)_{\text{air}}$ and $\mathrm{Ch}_2(\Upsilon_m)_{\text{air}}$. We see that the Hasse-Weil zeta functions of $\mathrm{Rep}_2(\Upsilon_m)_{\text{air}}$ and $\mathrm{Ch}_2(\Upsilon_m)_{\text{air}}$ satisfy functional equations.

The organization of this article is as follows: in §2, we review representations and molds on schemes. We also review the moduli of absolutely irreducible representations and the moduli of representations with Borel mold. In §3, we introduce several molds of degree 2: semi-simple mold, unipotent mold, unipotent mold over $\mathbb{F}_2$, and scalar mold. We also introduce the moduli of representations with scalar mold. In §4, we construct the moduli of equivalence classes of representations with semi-simple mold. In §5, we construct the moduli of equivalence classes of representations with unipotent mold over $\mathbb{Z}[1/2]$. In §6, we construct the moduli of equivalence classes of representations with unipotent mold over $\mathbb{F}_2$. In §7, we deal with different approach from §6. The approach in §7 gives us another construction of the moduli of equivalence classes of representations with unipotent mold over $\mathbb{F}_2$ by using derivations as in §5. In §8, we reformulate the moduli functors by using the notion of representations generating sheaves of $\mathcal{O}_X$-algebras which define molds of rank 2. In §9, we deal with discriminants which describe the absolutely irreducible representation part $\mathrm{Rep}_2(\Gamma)_{\mathrm{air}}$ in the representation variety $\mathrm{Rep}_2(\Gamma)$ as an appendix.

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2. Preliminaries

In this section, we review representations and molds on schemes. (For details, see [10] and [12].)

Definition 2.1 ([10])**.** Let Γ be a group or a monoid. By a *representation* of Γ on a scheme X, we understand a group homomorphism (or a monoid homomorphism) $\rho : \Gamma \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$. For two representations ρ and ρ', we say that ρ and ρ' are *equivalent* to each other (or $\rho \sim \rho'$) if there exists an $\mathcal{O}_X$-algebra isomorphism $\sigma : \mathrm{M}_n(\Gamma(X, \mathcal{O}_X)) \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$ such that $\sigma(\rho(\gamma)) = \rho'(\gamma)$ for each $\gamma \in \Gamma$.

Remark 2.2 ([10])**.** Let ρ and ρ' be n-dimensional representations of Γ on X. If $\rho \sim \rho'$, then for each $x \in X$ there exists $P \in \mathrm{GL}_n(\mathcal{O}_X(U))$

on a neighbourhood U of x such that $P^{-1}\rho(\gamma)P = \rho'(\gamma)$ on U for any $\gamma \in \Gamma$. Indeed, the group scheme PGL_n over $\mathbb{Z}$ represents the functor

$$\begin{array}{ccc} (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \mathrm{Aut}_{\mathcal{O}_X\text{-alg}}(\mathrm{M}_n(\mathcal{O}_X)). \end{array}$$

For details, see [10, Definition 6.1 and Theorem 6.2].

Definition 2.3 ([10])**.** Let Γ be a group or a monoid. The following contravariant functor is representable by an affine scheme:

$$\begin{array}{ccccc} \mathrm{Rep}_n(\Gamma) & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{\rho : \text{ rep. of deg } n \text{ for } \Gamma \text{ on } X\ \}. \end{array}$$

We call the affine scheme $\mathrm{Rep}_n(\Gamma)$ the *representation variety* of degree n for Γ. The group scheme PGL_n over $\mathbb{Z}$ acts on $\mathrm{Rep}_n(\Gamma)$ by $\rho \mapsto P^{-1}\rho P$. Each PGL_n-orbit forms an equivalence class of representations. If Γ is a finitely generated group (or monoid), then $\mathrm{Rep}_n(\Gamma)$ is of finite type over $\mathbb{Z}$.

Definition 2.4 ([12])**.** Let $\mathcal{A}$ be a subsheaf of $\mathrm{M}_n(\mathcal{O}_X)$ of $\mathcal{O}_X$-algebras on a scheme X. We say that $\mathcal{A}$ is a *mold* on X if $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ is locally free. Let $\mathrm{rank}\mathcal{A}$ denote the rank of a mold $\mathcal{A}$ as a locally free sheaf. For two molds $\mathcal{A}, \mathcal{B} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ on X, we say that $\mathcal{A}$ and $\mathcal{B}$ are *locally equivalent* if there exist an open covering $X = \cup_{i\in I} U_i$ and $P_i \in \mathrm{GL}_n(\mathcal{O}_X(U_i))$ such that $P_i(\mathcal{A}\mid_{U_i})P_i^{-1} = \mathcal{B}\mid_{U_i}$ for each $i \in I$.

Here let us introduce an example of molds.

Example 2.5 ([12])**.** We define the mold $\mathcal{B}_n$ on $\mathrm{Spec}\,\mathbb{Z}$ by

$$\mathcal{B}_n := \{(a_{ij}) \in \mathrm{M}_n(\mathbb{Z}) \mid a_{ij} = 0 \text{ for each } i > j\ \}.$$

For a mold $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ on a scheme X, we say that $\mathcal{A}$ is a *Borel mold* if $\mathcal{A}$ and $\mathcal{B}_n \otimes_{\mathbb{Z}} \mathcal{O}_X$ are locally equivalent to each other.

Definition 2.6 ([12])**.** Let $\mathcal{A}$ be a mold on a scheme X. For a representation ρ of Γ on X, we say that ρ has *mold type* $\mathcal{A}$ if the image $\rho(\Gamma)$ generates $\mathcal{A}$ as an $\mathcal{O}_X$-algebra.

Definition 2.7 ([12])**.** Let ρ be an n-dimensional representation of Γ on a scheme X. We say that ρ is an *absolutely irreducible representation* (or *air*) if ρ has mold type $\mathrm{M}_n(\mathcal{O}_X)$. We also say that ρ is a *representation with Borel mold* if ρ has a Borel mold type.

Proposition 2.8 ([10], [12])**.** *The contravariant functor*

$$\begin{array}{ccccc} \mathrm{Rep}_n(\Gamma)_{\mathrm{air}} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{\ \textit{air of degree } n \textit{ for } \Gamma \textit{ on } X\ \} \end{array}$$

is representable by an open subscheme $\mathrm{Rep}_n(\Gamma)_{\mathrm{air}}$ *of* $\mathrm{Rep}_n(\Gamma)$*. The contravariant functor*

$$\begin{array}{rcccl} \mathrm{Rep}_n(\Gamma)_B & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{l} \textit{rep. with Borel mold of degree } n \\ \textit{for } \Gamma \textit{ on } X \end{array} \right\} \end{array}$$

is representable by a subscheme $\mathrm{Rep}_n(\Gamma)_B$ *of* $\mathrm{Rep}_n(\Gamma)$*. The action of* PGL_n *on* $\mathrm{Rep}_n(\Gamma)$ *induces the ones of* PGL_n *on* $\mathrm{Rep}_n(\Gamma)_{\mathrm{air}}$ *and* $\mathrm{Rep}_n(\Gamma)_B$*.*

For absolutely irreducible representations, there exists a coarse moduli scheme.

Theorem 2.9 ([10])**.** *There exists a coarse moduli scheme* $\mathrm{Ch}_n(\Gamma)_{\mathrm{air}}$ *separated over* $\mathbb{Z}$ *associated to the following functor:*

$$\begin{array}{rcccl} \mathcal{E}q\mathcal{AIR}_n(\Gamma) & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{\rho : \textit{ air of degree } n \textit{ for } \Gamma \textit{ on } X\}/\sim . \end{array}$$

Furthermore, the canonical morphism $\mathrm{Rep}_n(\Gamma)_{\mathrm{air}} \to \mathrm{Ch}_n(\Gamma)_{\mathrm{air}}$ *gives a universal geometric quotient of* $\mathrm{Rep}_n(\Gamma)_{\mathrm{air}}$ *by* PGL_n*. If* Γ *is a finitely generated group (or monoid), then the moduli* $\mathrm{Ch}_n(\Gamma)_{\mathrm{air}}$ *is of finite type over* $\mathbb{Z}$*.*

For representations with Borel mold, there exists a fine moduli scheme.

Theorem 2.10 ([12])**.** *There exists a fine moduli scheme* $\mathrm{Ch}_n(\Gamma)_B$ *separated over* $\mathbb{Z}$ *associated to the sheafification* $\mathcal{E}q\mathcal{B}_n(\Gamma)$ *of the following functor with respect to Zariski topology:*

$$\begin{array}{ccl} (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \left\{ \begin{array}{l} \textit{rep. with Borel mold} \\ \textit{of degree } n \textit{ for } \Gamma \end{array} \right\} \Big/ \sim . \end{array}$$

Furthermore, the canonical morphism $\mathrm{Rep}_n(\Gamma)_B \to \mathrm{Ch}_n(\Gamma)_B$ *gives a universal geometric quotient of* $\mathrm{Rep}_n(\Gamma)_B$ *by* PGL_n*. If* Γ *is a finitely generated group (or monoid), then the moduli is of finite type over* $\mathbb{Z}$*.*

3. The degree 2 case

From now on, we deal mainly with the degree 2 case.

Let $A_2(\Gamma)$ be the coordinate ring of the representation variety of degree 2 for a group or a monoid Γ. Let $\sigma_\Gamma : \Gamma \to \mathrm{M}_2(A_2(\Gamma))$ be the universal representation of degree 2 for Γ.

Definition 3.1. Let $A_2(\Gamma)^{\mathrm{Ch}}$ be the subalgebra of $A_2(\Gamma)$ generated by $\{\mathrm{tr}(\sigma_\Gamma(\gamma)), \det(\sigma_\Gamma(\gamma)) \mid \gamma \in \Gamma\}$ over $\mathbb{Z}$. We denote $\mathrm{Spec}A_2(\Gamma)^{\mathrm{Ch}}$ by $\mathrm{Ch}_2(\Gamma)$.

In [12, Example 1.3] we investigated the moduli of molds:

$$\begin{aligned} \mathrm{Mold}_{2,1} &= \mathrm{Spec}\,\mathbb{Z}, \\ \mathrm{Mold}_{2,2} &= \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{2,3} &= \mathbb{P}^1_{\mathbb{Z}}, \\ \mathrm{Mold}_{2,4} &= \mathrm{Spec}\,\mathbb{Z}. \end{aligned}$$

Let k be an algebraically closed field, for simplicity. Let us classify k-subalgebras A of $\mathrm{M}_2(k)$ up to inner automorphisms of $\mathrm{M}_2(k)$ for explaining molds of degree 2. In the case $\dim A = 4$, A is equal to $\mathrm{M}_2(k)$. For any subalgebra A of dimension 3, there exists $P \in \mathrm{GL}_2(k)$ such that $P^{-1}AP = \mathcal{B}_2(k) := \{(a_{ij}) \in \mathrm{M}_2(k) \mid a_{21} = 0\}$. In the case $\dim A = 2$, there exists $X \in A$ such that $A = kI_2 + kX$. For $[X] \in \mathrm{M}_2(k)/kI_2$, we can define a mold $A = kI_2 + kX$, which is independent from choosing a representative $X \in \mathrm{M}_2(k)$ of $[X]$. This is the reason why $\mathrm{Mold}_{2,2}(k) \cong \mathbb{P}_*(\mathrm{M}_2(k)/kI_2) = \mathbb{P}^2_k$. There exist two types of molds of rank 2. The one is a semi-simple algebra, and the other is a non-semi-simple algebra. In other words, the former is $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a, b \in k \right\}$, and the latter is $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a, b \in k \right\}$ up to inner automorphisms. Of course, a subalgebra A of dimension 1 is equal to kI_2.

By using the classification of k-subalgebras of $\mathrm{M}_2(k)$, we introduce several molds of degree 2. For the case of rank 4, we consider the full matrix ring mold $\mathrm{M}_2(\mathcal{O}_X)$. For the case of rank 3, we introduced Borel molds.

Here we introduce several types of molds of rank 2. There are two types of molds of rank 2: the semi-simple subalgebra case and the non-semi-simple subalgebra case. Moreover we can divide the non-semi-simple 2-dimensional subalgebra case into two types: the $\mathrm{ch} \neq 2$ type and the $\mathrm{ch} = 2$ type.

Notation 3.2. Let R be a commutative ring. For $X \in \mathrm{M}_2(R)$, we denote $\mathrm{tr}(X)^2 - 4\det(X)$ by $m(X)$. Remark that $m(X) = 2\mathrm{tr}(X^2) - (\mathrm{tr}(X))^2$.

Remark 3.3. For $X \in \mathrm{M}_2(R)$, $m(X)$ is the discriminant of the characteristic polynomial of X. If R is a field, then $m(X) \neq 0$ if and only if X is semi-simple and not scalar.

Definition 3.4. Let X be a scheme. Let $\mathcal{A} \subseteq \mathrm{M}_2(\mathcal{O}_X)$ be a rank 2 mold on X. We say that $\mathcal{A}$ is *semi-simple* if there exists $P_x \in \mathcal{A}_x$ such that $m(P_x) \not\equiv 0$ in the residue field $k(x)$ for each $x \in X$.

Definition 3.5. Let X be a scheme over $\mathbb{Z}[1/2]$. Let $\mathcal{A} \subseteq \mathrm{M}_2(\mathcal{O}_X)$ be a rank 2 mold on X. We say that $\mathcal{A}$ is *unipotent* if $m(A) = 0$ for each $A \in \mathcal{A}(U)$ and for each open set $U \subseteq X$.

Definition 3.6. Let X be a scheme over $\mathbb{F}_2$. Let $\mathcal{A} \subseteq \mathrm{M}_2(\mathcal{O}_X)$ be a rank 2 mold on X. We say that $\mathcal{A}$ is *unipotent over* $\mathbb{F}_2$ if $\mathrm{tr}(A) = 0$ for each $A \in \mathcal{A}(U)$ and for each open set $U \subseteq X$.

Remark 3.7. The name "unipotent" seems to be strange. However, the author calls non-semi-simple molds of rank 2 unipotent molds because each unipotent mold over an algebraically closed field k is generated by a unipotent matrix of $\mathrm{M}_2(k)$.

For each type of molds of rank 2, we introduce representations with a given mold.

Definition 3.8. For a 2-dimensional representation ρ for a group or a monoid Γ on a scheme X, we say that ρ is a *representation with semi-simple mold* if $\mathcal{O}_X[\rho(\Gamma)]$ is a semi-simple mold on X. When X is a scheme over $\mathbb{Z}[1/2]$ (or over $\mathbb{F}_2$), we say that ρ is a *representation with unipotent mold* (or *unipotent mold over* $\mathbb{F}_2$) if $\mathcal{O}_X[\rho(\Gamma)]$ is a unipotent mold (or a unipotent mold over $\mathbb{F}_2$, respectively) on X.

For each case of molds of rank 2, we construct the moduli of representations in §4-§6.

Finally, we consider molds of rank 1. This case is trivial. Indeed, any mold of rank 1 is the mold consisting of scalar matrices. Let us introduce the following definition for any degree.

Definition 3.9. Let X be a scheme. We say that $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ is a *scalar mold* if $\mathcal{A}$ is a rank 1 mold on X. In other words, $\mathcal{A}$ is a scalar mold if and only if $\mathcal{A} = \mathcal{O}_X \cdot I_n$.

Definition 3.10. For an n-dimensional representation ρ for a group or a monoid Γ on a scheme X, we say that ρ is a *representation with scalar mold* if $\mathcal{O}_X[\rho(\Gamma)]$ is a scalar mold on X.

Proposition 3.11. *The contravariant functor*

$$\begin{array}{rccl} \mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}} & : & (\mathbf{Sch})^{op} & \rightarrow & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{l} \textit{rep. with scalar mold} \\ \textit{of degree } n \textit{ for } \Gamma \textit{ on } X \end{array} \right\} \end{array}$$

is representable by a closed subscheme $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}}$ *of* $\mathrm{Rep}_n(\Gamma)$*. The induced action of* PGL_n *on* $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}}$ *is trivial.*

Proof. Let $A_n(\Gamma)$ be the coordinate ring of the representation variety $\mathrm{Rep}_n(\Gamma)$. Let $\sigma_\Gamma : \Gamma \to \mathrm{M}_n(A_n(\Gamma))$ be the universal representation of degree n for Γ. We denote by I the ideal of $A_n(\Gamma)$ generated by $\{\sigma_\Gamma(\gamma)_{ij} \mid 1 \leq i \neq j \leq n, \gamma \in \Gamma\} \cup \{\sigma_\Gamma(\gamma)_{ii} - \sigma_\Gamma(\gamma)_{jj} \mid 1 \leq i < j \leq n, \gamma \in \Gamma\}$. Then it is easy to check that $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}}$ is representable by the affine scheme $\mathrm{Spec} A_n(\Gamma)/I$. Since I is PGL_n-invariant and the action of PGL_n on $A_n(\Gamma)/I$ is trivial, the induced action of PGL_n on $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}}$ is trivial. $\square$

Theorem 3.12. *There exists a fine moduli scheme* $\mathrm{Ch}_n(\Gamma)_{\mathrm{scalar}}$ *separated over* $\mathbb{Z}$ *associated to the following contravariant functor:*

$$\begin{array}{rcccl} \mathcal{E}q\mathcal{S}_n(\Gamma) & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{l} \textit{rep. with scalar mold} \\ \textit{of degree } n \textit{ for } \Gamma \textit{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

The moduli $\mathrm{Ch}_n(\Gamma)_{\mathrm{scalar}}$ *is isomorphic to* $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}}$*. Moreover, they are isomorphic to* $\mathrm{Rep}_1(\Gamma) \cong \mathrm{Ch}_1(\Gamma) := \mathrm{Rep}_1(\Gamma)/\mathrm{PGL}_1$*. In particular, if* Γ *is a finitely generated group (or monoid), then the moduli is of finite type over* $\mathbb{Z}$*.*

Proof. Since the action of PGL_n on $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}}$ is trivial, the affine scheme $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}}$ also represents the functor $\mathcal{E}q\mathcal{S}_n(\Gamma)$. We easily see that $A_n(\Gamma)/I \cong A_1(\Gamma)$, where I is defined in the proof of Proposition 3.11. The action of $\mathrm{PGL}_1 \cong \mathrm{Spec}\,\mathbb{Z}$ on $\mathrm{Rep}_1(\Gamma)$ is trivial. Hence we see that $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}} \cong \mathrm{Rep}_1(\Gamma) \cong \mathrm{Ch}_1(\Gamma)$. If Γ is finitely generated, then $\mathrm{Rep}_n(\Gamma)$ is of finite type over $\mathbb{Z}$, and therefore so is $\mathrm{Rep}_n(\Gamma)_{\mathrm{scalar}}$. $\square$

4. Semi-simple mold

In §4-§6, we only deal with rank 2 molds of degree 2. In this section, we investigate the semi-simple mold case.

Definition 4.1. Let $\sigma_\Gamma : \Gamma \to \mathrm{M}_2(A_2(\Gamma))$ be the universal representation of degree 2 for a group or a monoid Γ. For $\alpha, \beta, \gamma \in \Gamma$, we define the matrix $M(\alpha,\beta,\gamma)$ by

$$M(\alpha,\beta,\gamma) := \begin{pmatrix} \sigma_\Gamma(\alpha)_{11} & \sigma_\Gamma(\beta)_{11} & \sigma_\Gamma(\gamma)_{11} \\ \sigma_\Gamma(\alpha)_{12} & \sigma_\Gamma(\beta)_{12} & \sigma_\Gamma(\gamma)_{12} \\ \sigma_\Gamma(\alpha)_{21} & \sigma_\Gamma(\beta)_{21} & \sigma_\Gamma(\gamma)_{21} \\ \sigma_\Gamma(\alpha)_{22} & \sigma_\Gamma(\beta)_{22} & \sigma_\Gamma(\gamma)_{22} \end{pmatrix}.$$

We define the closed subscheme $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\le 2}$ of $\mathrm{Rep}_2(\Gamma)$ by

$$\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\le 2} := \left\{ \rho \in \mathrm{Rep}_2(\Gamma) \,\middle|\, \begin{array}{c} \text{all } (3\times 3) \text{ minor determinants of} \\ M(\alpha,\beta,\gamma) \text{ are } 0 \text{ for each } \alpha,\beta,\gamma\in\Gamma \end{array} \right\}.$$

We also define the open subscheme $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}2}$ of the affine scheme $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\le 2}$ by

$$\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}2} := \{\rho\in\mathrm{Rep}_2(\Gamma) \mid \mathcal{O}_X[\rho(\Gamma)] \text{ is a rank 2 mold }\}.$$

Definition 4.2. We define the *representation variety with semi-simple mold* of degree 2 for a group or a monoid Γ by

$$\begin{array}{rcccl} \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \{\rho\in\mathrm{Rep}_2(\Gamma) \mid \rho \text{ has a semi-simple mold }\}. \end{array}$$

We easily see that $\mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}}$ is an open subscheme of $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}2}$.

Remark 4.3. The scheme $\mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}}$ is an open subscheme of the affine scheme $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\le 2}$ where $m(\sigma_\Gamma(\gamma))$ does not vanish for some $\gamma\in\Gamma$ by Remark 3.3. Here recall that $m(\sigma_\Gamma(\gamma)) = \mathrm{tr}(\sigma_\Gamma(\gamma))^2 - 4\det(\sigma_\Gamma(\gamma))$.

Let us denote by $A_2(\Gamma)_{\mathrm{rk}\le 2}$ the coordinate ring of the affine scheme $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\le 2}$. We define $A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\le 2}$ as the subring of $A_2(\Gamma)_{\mathrm{rk}\le 2}$ generated by $\{\mathrm{tr}(\sigma_\Gamma(\gamma)), \det(\sigma_\Gamma(\gamma)) \mid \gamma\in\Gamma\}$ over $\mathbb{Z}$. We also denote by $\mathrm{Ch}_2(\Gamma)_{\mathrm{rk}\le 2}$ the spectrum of $A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\le 2}$. We define the open subscheme $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ of $\mathrm{Ch}_2(\Gamma)_{\mathrm{rk}\le 2}$ by

$$\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}} := \bigcup_{\gamma\in\Gamma} \mathrm{Spec}\ (A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\le 2})_{m(\sigma_\Gamma(\gamma))}.$$

Then we have the canonical morphism

$$\pi_{\Gamma,\mathrm{s.s.}} : \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}.$$

For $\gamma\in\Gamma$ we define

$$\begin{array}{rcl} \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma} & := & \{x\in\mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}} \mid m(\sigma_\Gamma(\gamma)) \not\equiv 0 \text{ in } k(x)\} \\ & = & \mathrm{Spec}\ (A_2(\Gamma)_{\mathrm{rk}\le 2})_{m(\sigma_\Gamma(\gamma))} \end{array}$$

and

$$\begin{array}{rcl} \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma} & := & \{x\in\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}} \mid m(\sigma_\Gamma(\gamma)) \not\equiv 0 \text{ in } k(x)\} \\ & = & \mathrm{Spec}\ (A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\le 2})_{m(\sigma_\Gamma(\gamma))}. \end{array}$$

Then we have the canonical morphism

$$\pi_{\Gamma,\mathrm{s.s.},\gamma} : \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma} \to \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}.$$

For a group or a monoid Γ, we have the following diagram for each $\gamma \in \Gamma$:

$$\begin{array}{ccccc} \mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\leq 2} & \supseteq & \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}} & \supseteq & \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ch}_2(\Gamma)_{\mathrm{rk}\leq 2} & \supseteq & \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}} & \supseteq & \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}. \end{array}$$

Proposition 4.4. *If Γ is a finitely generated group or monoid, then $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\leq 2}$ and $\mathrm{Ch}_2(\Gamma)_{\mathrm{rk}\leq 2}$ are of finite type over $\mathbb{Z}$.*

Proof. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a set of generators of Γ. We may assume that α_i^{-1} is also an element of S for each $1 \leq i \leq n$ if Γ is a group. The coordinate ring $A_2(\Gamma)_{\mathrm{rk}\leq 2}$ is generated by all entries of $\sigma_\Gamma(\alpha_i)$ for $1 \leq i \leq n$ over $\mathbb{Z}$. Hence $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\leq 2}$ is of finite type over $\mathbb{Z}$. Let $A_i := \sigma_\Gamma(\alpha_i)$ for $1 \leq i \leq n$. Then the coordinate ring $A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2}$ is generated by $\{\det(A_i) \mid 1 \leq i \leq n\}$ and $\{\mathrm{tr}(A_{i_1}A_{i_2}\cdots A_{i_k}) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ over $\mathbb{Z}$. Indeed, we can verify it by using the following equalities:

$$\begin{aligned} \mathrm{tr}(X^2Y) &= \mathrm{tr}(X)\mathrm{tr}(XY) - \det(X)\mathrm{tr}(Y) \\ \mathrm{tr}(XYZ) &= -\mathrm{tr}(XZY) + \mathrm{tr}(X)\mathrm{tr}(YZ) + \mathrm{tr}(Y)\mathrm{tr}(ZX) \\ &\quad +\mathrm{tr}(Z)\mathrm{tr}(YX) - \mathrm{tr}(X)\mathrm{tr}(Y)\mathrm{tr}(Z) \end{aligned}$$

for 2×2 matrices X, Y, Z. These equalities have been well known (For proofs see [15] or [11, Appendix]). Therefore $\mathrm{Ch}_2(\Gamma)_{\mathrm{rk}\leq 2}$ is of finite type over $\mathbb{Z}$. $\square$

Definition 4.5. Let $\Upsilon_1 = \langle \alpha \rangle$ be the free monoid of rank 1. We call the morphism $\pi_{\Upsilon_1,\mathrm{s.s.},\alpha} : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.},\alpha} \to \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.},\alpha}$ the *prototype* with semi-simple mold of degree 2.

Let $\mathrm{F}_1 = \langle \alpha \rangle$ be the free group of rank 1. We call the morphism $\pi_{\mathrm{F}_1,\mathrm{s.s.},\alpha} : \mathrm{Rep}_2(\mathrm{F}_1)_{\mathrm{s.s.},\alpha} \to \mathrm{Ch}_2(\mathrm{F}_1)_{\mathrm{s.s.},\alpha}$ the *prototype for group representations* with semi-simple mold of degree 2.

Let σ_{Υ_1} be the universal representation of degree 2 for Υ_1. Put $\sigma_{\Upsilon_1}(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we see that the coordinate ring $A_2(\Upsilon_1)$ of $\mathrm{Rep}_2(\Upsilon_1)$ is isomorphic to the polynomial ring $\mathbb{Z}[a, b, c, d]$. Note that $\mathrm{Rep}_2(\Upsilon_1) = \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}\leq 2}$ and that $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} = D(a-d) \cup D(b) \cup D(c) \subseteq \mathrm{Rep}_2(\Upsilon_1) = \mathrm{Spec}\,\mathbb{Z}[a, b, c, d]$.

Put $D := ad - bc$ and $T := a + d$. Let $A_2(\Upsilon_1)^{\mathrm{Ch}}$ be the subalgebra of $A_2(\Upsilon_1)$ generated by $\{\mathrm{tr}(\sigma_{\Upsilon_1}(\gamma)), \det(\sigma_{\Upsilon_1}(\gamma)) \mid \gamma \in \Upsilon_1\}$ over $\mathbb{Z}$. Then $A_2(\Upsilon_1)^{\mathrm{Ch}}$ is isomorphic to the polynomial ring $\mathbb{Z}[T, D]$. Set $\mathrm{Ch}_2(\Upsilon_1) := \mathrm{Spec}A_2(\Upsilon_1)^{\mathrm{Ch}}$. Then $\mathrm{Ch}_2(\Upsilon_1) = \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{rk}\leq 2}$.

Proposition 4.6. *Let R be a commutative ring. Let $A \in \mathrm{M}_2(R)$. For each $n \in \mathbb{N}$, there exists $f(x,y) \in \mathbb{Z}[x,y]$ such that $m(A^n) = m(A)f(\mathrm{tr}A, \det A)$.*

Proof. Let us claim that

$$m(A^n) =$$
$$\begin{cases} m(A) \cdot \left[\displaystyle\sum_{k=0}^{(n-3)/2} \det(A)^k \mathrm{tr}(A^{n-2k-1}) + \det(A)^{(n-1)/2} \right]^2 & (n: \text{ odd }) \\ m(A) \cdot \left[\displaystyle\sum_{k=0}^{(n-2)/2} \det(A)^k \mathrm{tr}(A^{n-2k-1}) \right]^2 & (n: \text{ even }). \end{cases}$$

Since $\mathrm{tr}(A^k)$ can be expressed by a polynomial in $\mathbb{Z}[\mathrm{tr}(A), \det(A)]$ for each $k \in \mathbb{N}$, the statement follows from this claim. It only suffices to prove that this claim holds for $A = \sigma_{\Upsilon_1}(\alpha) \in \mathrm{M}_2(A_2(\Upsilon_1))$.

For $A \in \mathrm{M}_2(k)$ with an algebraically closed field k, let λ, μ be eigenvalues of A. Note that $m(A) = (\lambda - \mu)^2$ and $m(A^n) = (\lambda^n - \mu^n)^2$. Then

$$\begin{aligned} m(A^n) &= (\lambda - \mu)^2 (\lambda^{n-1} + \lambda^{n-2}\mu + \cdots + \lambda\mu^{n-2} + \mu^{n-1})^2 \\ &= m(A)\{\mathrm{tr}(A^{n-1}) + \det(A)\mathrm{tr}(A^{n-3}) + \det(A)^2 \mathrm{tr}(A^{n-5}) + \cdots\}^2. \end{aligned}$$

Hence the claim holds for $A \in \mathrm{M}_2(k)$ with $k = \overline{k}$. Because the claim holds for an algebraic closure k of the quotient field $Q(A_2(\Upsilon_1))$ of $A_2(\Upsilon_1)$, it also holds for $Q(A_2(\Upsilon_1))$ and for $A_2(\Upsilon_1)$. This completes the proof. $\square$

Remark 4.7. Using Proposition 4.6, we easily see that $\mathrm{Rep}_2(\Upsilon_1)_{\text{s.s.}} = \mathrm{Rep}_2(\Upsilon_1)_{\text{s.s.},\alpha}$ and that $\mathrm{Ch}_2(\Upsilon_1)_{\text{s.s.}} = \mathrm{Ch}_2(\Upsilon_1)_{\text{s.s.},\alpha}$. Note that $m(A^{-1}) = m(A)(\det(A))^{-2}$ for $A \in \mathrm{GL}_2(R)$ with a commutative ring R. Hence we also see that $\mathrm{Rep}_2(\mathrm{F}_1)_{\text{s.s.}} = \mathrm{Rep}_2(\mathrm{F}_1)_{\text{s.s.},\alpha}$ and that $\mathrm{Ch}_2(\mathrm{F}_1)_{\text{s.s.}} = \mathrm{Ch}_2(\mathrm{F}_1)_{\text{s.s.},\alpha}$.

We have the following diagram for the free monoid $\Upsilon_1 = \langle \alpha \rangle$:

$$\begin{array}{ccccccc} \mathrm{Rep}_2(\Upsilon_1) & & & & & & \\ \| & & & & & & \\ \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}\le 2} & \supset & \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} & \supset & \mathrm{Rep}_2(\Upsilon_1)_{\text{s.s.}} & = & \mathrm{Rep}_2(\Upsilon_1)_{\text{s.s.},\alpha} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ch}_2(\Upsilon_1) & = & \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{rk}\le 2} & \supset & \mathrm{Ch}_2(\Upsilon_1)_{\text{s.s.}} & = & \mathrm{Ch}_2(\Upsilon_1)_{\text{s.s.},\alpha}. \end{array}$$

Put $m := T^2 - 4D$. The morphism $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ is given by $D(a-d) \cup D(b) \cup D(c) \to \mathrm{Spec}\mathbb{Z}[T, D]$. The prototype

$\pi_{\Upsilon_1,\mathrm{s.s.}} : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.}}$ is induced by the ring homomorphism $\mathbb{Z}[T, D]_m \to \mathbb{Z}[a, b, c, d]_m$.

Lemma 4.8. *Let (R, m) be a local ring. Let $A \in \mathrm{M}_2(R)$. Suppose that $(A \mod m)$ is not a scalar matrix of $\mathrm{M}_2(R/m)$. Then there exists $P \in \mathrm{GL}_2(R)$ such that*

$$P^{-1}AP = \begin{pmatrix} 0 & -\det(A) \\ 1 & \mathrm{tr}(A) \end{pmatrix}.$$

If $Q \in \mathrm{M}_2(R)$ satisfies $AQ = QA$, then $Q = \lambda I_2 + \mu A$ for some $\lambda, \mu \in R$.

Proof. Put $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From the assumption, at least one of $a - d$, b, c is contained in $R^\times$. Assume that $b \in R^\times$. Then the vectors $e_2 := {}^t(0, 1)$ and $Ae_2 \in R^2$ form a basis of R^2. With respect to the basis $\{e_2, Ae_2\}$, the linear map $A : R^2 \to R^2$ can be expressed as $\begin{pmatrix} 0 & -\det(A) \\ 1 & \mathrm{tr}(A) \end{pmatrix}$. In the case $c \in R^\times$, we can choose $\{e_1, Ae_1\}$ as a basis of R^2, where $e_1 := {}^t(1, 0)$. Then we can change A into the form which we want. If $a - d \in R^\times$ and $b, c \notin R^\times$, then the vectors $e_1 + e_2 = {}^t(1, 1)$ and $A(e_1 + e_2)$ form a basis of R^2. Similarly we can change A into the desired form.

To prove the latter part of the statement, we may assume that $A = \begin{pmatrix} 0 & -\det(A) \\ 1 & \mathrm{tr}(A) \end{pmatrix}$. By direct calculation, we see that $AQ = QA$ implies $Q = \lambda I_2 + \mu A$ for some $\lambda, \mu \in R$. □

Proposition 4.9. *The morphism $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ is smooth and surjective. In particular, it is faithfully flat.*

Proof. Let I be an ideal of a local ring R with $I^2 = 0$. For a given commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} & \to & \mathrm{Ch}_2(\Upsilon_1) \\ \uparrow & & \uparrow \\ \mathrm{Spec} R/I & \to & \mathrm{Spec} R, \end{array}$$

we obtain $(T, D) \in R^2$ and $\overline{A} \in \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}(R/I) \subset \mathrm{M}_2(R/I)$ such that $\mathrm{tr}(\overline{A}) \equiv T$ and $\det(\overline{A}) \equiv D \pmod I$. By Lemma 4.8, there exists $\overline{P} \in \mathrm{GL}_2(R/I)$ such that

$$\overline{P}^{-1}\,\overline{A}\,\overline{P} \equiv B := \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix} \pmod I.$$

Let us take $P \in \mathrm{GL}_2(R)$ such that $P \equiv \overline{P} \pmod{I}$. Put $A = PBP^{-1}$. Then $A \in \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}(R)$ such that $\mathrm{tr}(A) = T$ and $\det(A) = D$. Hence we obtain a morphism $\mathrm{Spec}R \to \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}$ satisfying the commutativity. This implies that the morphism $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ is smooth. Surjectivity follows from that we can take such matrix as B above for a given k-valued point $(T, D) \in \mathrm{Ch}_2(\Upsilon_1)(k)$ with a field k. Since smoothness implies flatness, it is faithfully flat. $\square$

Lemma 4.10. *Let R be a commutative ring. For $X, Y \in \mathrm{M}_2(R)$ and $a, b \in R$, we have*

$$\det(aX + bY) = a^2 \det(X) + b^2 \det(Y) + ab(\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(XY)).$$

Proof. By direct calculation, we can check the formula above. $\square$

Lemma 4.11. *Let (R, m, k) be an artinian local ring, and I be an ideal of R with $mI = 0$. For $A \in \mathrm{M}_2(R)$, let us define the k-linear map $[A, -] : \mathrm{M}_2(I) \to \mathrm{M}_2(I)$ by $X \mapsto AX - XA$. If $(A \mod m)$ is not a scalar matrix of $\mathrm{M}_2(k)$, then*

$$\mathrm{Im}[A, -] = \{Y \in \mathrm{M}_2(I) \mid \mathrm{tr}(Y) = \mathrm{tr}(AY) = 0\}.$$

Proof. Since $mI = 0$, we can regard I as a vector space over $R/m = k$. Put $d := \dim_k I < \infty$. Set $N := \{Y \in \mathrm{M}_2(I) \mid \mathrm{tr}(Y) = \mathrm{tr}(AY) = 0\}$. If $Y = [A, X] \in \mathrm{Im}[A, -]$, then $\mathrm{tr}(Y) = \mathrm{tr}(AX) - \mathrm{tr}(XA) = 0$ and $\mathrm{tr}(AY) = \mathrm{tr}(AAX) - \mathrm{tr}(AXA) = 0$. Hence $\mathrm{Im}[A, -] \subseteq N$. For showing that $\mathrm{Im}[A, -] = N$, we prove that the dimensions of the both sides coincide. In order to calculate the dimensions, we may change A into

$$P^{-1}AP = \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix}$$

for suitable $P \in \mathrm{GL}_2(R)$ by considering the automorphism $\mathrm{Ad}(P) : \mathrm{M}_2(I) \to \mathrm{M}_2(I)$ by $X \mapsto P^{-1}XP$.

If $X \in \mathrm{Ker}[A, -]$, then $X = \lambda I_2 + \mu A$ for some $\lambda, \mu \in R$ by Lemma 4.8. Since $X \in \mathrm{M}_2(I)$, we get $\lambda, \mu \in I$. Hence we see that $\dim_k \mathrm{Ker}[A, -] = \dim_k(I \cdot I_2 + I \cdot A) = 2d$ and that $\dim_k \mathrm{Im}[A, -] = \dim_k \mathrm{M}_2(I) - \dim_k \mathrm{Ker}[A, -] = 2d$. On the other hand, if $X \in N$, then $\mathrm{tr}(X) = \mathrm{tr}(AX) = 0$. By direct calculation, we have $\dim_k N = 2d$. Thus we have proved that $\dim_k \mathrm{Im}[A, -] = \dim_k N = 2d$ and that $\mathrm{Im}[A, -] = N$. $\square$

Let $s : \mathrm{Ch}_2(\Upsilon_1) \to \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}$ by $(T, D) \mapsto \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix}$. Then $\pi \circ s = 1_{\mathrm{Ch}_2(\Upsilon_1)}$. We have the following proposition:

Proposition 4.12. *The composition of the morphisms*

$$\begin{array}{ccccc} \mathrm{Ch}_2(\Upsilon_1) \times \mathrm{PGL}_2 & \stackrel{(s,id)}{\to} & \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \times \mathrm{PGL}_2 & \stackrel{\sigma}{\to} & \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \\ ((T,D),P) & \mapsto & \left(\begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix}, P \right) & \mapsto & P^{-1} \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix} P \end{array}$$

is smooth and surjective. In particular, it is faithfully flat.

Proof. Let (R, m, k) be an artinian local ring, and let I be an ideal of R with $mI = 0$. For a given commutative diagram

$$\begin{array}{ccc} \mathrm{Ch}_2(\Upsilon_1) \times \mathrm{PGL}_2 & \stackrel{\sigma\circ(s,id)}{\to} & \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \\ \uparrow & & \uparrow \\ \mathrm{Spec}\, R/I & \to & \mathrm{Spec}\, R, \end{array}$$

we obtain $A \in \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}}(R)$, $(\overline{T}, \overline{D}) \in \mathrm{Ch}_2(\Upsilon_1)(R/I)$, and $\overline{P} \in \mathrm{GL}_2(R/I)$ such that $\overline{P}^{-1} \begin{pmatrix} 0 & -\overline{D} \\ 1 & \overline{T} \end{pmatrix} \overline{P} \equiv A$ in $\mathrm{M}_2(R/I)$. Take $P \in \mathrm{GL}_2(R)$ such that $(P \mod I) = \overline{P}$. Put $D := \det A$ and $T := \mathrm{tr}(A)$. Note that $(D \mod I) = \overline{D}$ and $(T \mod I) = \overline{T}$.

Set $Y := P^{-1} \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix} P - A \in \mathrm{M}_2(I)$. Let us show that $\mathrm{tr}(AY) = 0$. Remark that $\det Y = 0$ by $I^2 = 0$ and that $\mathrm{tr}(Y) = 0$. Using Lemma 4.10, we have

$$\begin{aligned} D &= \det(P^{-1} \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix} P) \\ &= \det(A + Y) \\ &= \det A + \det Y + \mathrm{tr}(A)\mathrm{tr}(Y) - \mathrm{tr}(AY) \\ &= D - \mathrm{tr}(AY). \end{aligned}$$

Hence we have proved $\mathrm{tr}(AY) = 0$.

By Lemma 4.11, we have $Y \in \mathrm{Im}[A, -]$. There exists $X \in \mathrm{M}_2(I)$ such that $[A, X] = Y$. Put $P' := P(I_2 - X) \in \mathrm{GL}_2(R)$. Then $P'^{-1} \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix} P' = (I_2 + X)(A + Y)(I_2 - X) = A + Y - [A, X] = A$.

Now let us define the morphism $\mathrm{Spec} R \to \mathrm{Ch}_2(\Upsilon_1) \times \mathrm{PGL}_2$ corresponding to $((T, D), P')$. Verifying the commutativity, we see that the morphism is smooth. By Lemma 4.8 we see that the morphism is surjective. Hence it is faithfully flat. $\square$

Proposition 4.13. *The morphism*

$$\begin{array}{ccc} \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \times \mathrm{PGL}_2 & \to & \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \times_{\mathrm{Ch}_2(\Upsilon_1)} \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \\ (\rho, P) & \mapsto & (\rho, P^{-1}\rho P) \end{array}$$

is smooth and surjective. In particular, it is faithfully flat.

Proof. Let (R, m, k) be an artinian local ring, and let I be an ideal of R with $mI = 0$. For a given commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \times \mathrm{PGL}_2 & \to & \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \times_{\mathrm{Ch}_2(\Upsilon_1)} \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \\ \uparrow & & \uparrow \\ \mathrm{Spec}\, R/I & \to & \mathrm{Spec}\, R, \end{array}$$

we obtain $(A, B) \in \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}(R) \times_{\mathrm{Ch}_2(\Upsilon_1)(R)} \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}(R)$ and $\overline{P} \in \mathrm{GL}_2(R/I)$ such that $\overline{P}^{-1} A \overline{P} \equiv B$ in $\mathrm{M}_2(R/I)$. For proving that the morphism is smooth, we define a morphism $\mathrm{Spec} R \to \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \times \mathrm{PGL}_2$ satisfying the commutativity. Put $T = \mathrm{tr}(A) = \mathrm{tr}(B)$ and $D = \det A = \det B$. Let us take $P \in \mathrm{GL}_2(R)$ such that $(P \mod I) = \overline{P}$. Then set $C := P^{-1}AP - B \in \mathrm{M}_2(I)$.

Let us show that $\mathrm{tr}(C) = \mathrm{tr}(BC) = 0$. Indeed, $\mathrm{tr}(C) = \mathrm{tr}(P^{-1}AP) - \mathrm{tr}(B) = T - T = 0$. Note that $\det C = 0$ by $I^2 = 0$. Using Lemma 4.10, we have

$$\begin{aligned} D = \det(P^{-1}AP) &= \det(B + C) \\ &= \det B + \det C + \mathrm{tr}(B)\mathrm{tr}(C) - \mathrm{tr}(BC) \\ &= D - \mathrm{tr}(BC). \end{aligned}$$

Hence we have verified $\mathrm{tr}(BC) = 0$.

By Lemma 4.11, we have $C \in \mathrm{Im}[B, -]$. There exists $X \in \mathrm{M}_2(I)$ such that $[B, X] = C$. Put $P' := P(I_2 - X) \in \mathrm{GL}_2(R)$. Then $P'^{-1}AP' = (I_2 + X)(B + C)(I_2 - X) = B + C - [B, X] = B$. Now let us define the morphism $\mathrm{Spec} R \to \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \times \mathrm{PGL}_2$ corresponding to (A, P'). Since $P'^{-1}AP' = B$, we can verify the commutativity. Therefore the morphism is smooth.

By Lemma 4.8 we see that the morphism is surjective. Hence it is faithfully flat. □

Let us introduce the following two lemmas on sufficient conditions for a given morphism to be a universal geometric quotient (For the definition of universal geometric quotient, see [9]).

Lemma 4.14. *Let G be an affine group scheme over an affine scheme S. Assume that the S-morphism $\sigma : G \times_S X \to X$ is a group action of G on an S-scheme X and that the action of G on an S-scheme Y is trivial. Let $\pi : X \to Y$ be an affine G-equivariant faithfully flat locally of finite presentation S-morphism. If the morphism $(\sigma, p_2) : G \times_S X \to X \times_Y X$ is faithfully flat, then $\pi : X \to Y$ is a universal geometric quotient by G.*

Proof. From the assumption, we have $\pi \circ \sigma = \pi \circ p_2 : G \times_S X \to Y$. We also see that π is surjective and that the image of the morphism $(\sigma, p_2) : G \times_S X \to X \times_S X$ is $X \times_Y X$. Since π is faithfully flat locally of finite presentation, it is universally open (*cf.* [4, Theorem 2.4.6]). For verifying $\pi_*(\mathcal{O}_X)^G = \mathcal{O}_Y$, we only need to check that $\mathcal{O}_X(\pi^{-1}(U))^G = \mathcal{O}_Y(U)$ for each affine open subscheme U of Y. Set $U = \mathrm{Spec} A$ and $\pi^{-1}(U) = \mathrm{Spec} B$. Because $A \to B$ is faithfully flat, $0 \to A \to B \overset{\phi_1 - \phi_2}{\to} B \otimes_A B$ is exact, where $\phi_1(b) = b \otimes 1$ and $\phi_2(b) = 1 \otimes b$ for $b \in B$ (for example, see [8, Proposition 2.18]). Put $S = \mathrm{Spec}\, C$ and $G = \mathrm{Spec} R$. The ring homomorphism $B \otimes_A B \to R \otimes_C B$ induced by $(\sigma, p_2) : G \times_S X \to X \times_Y X$ is faithfully flat. Since $B \otimes_A B \to R \otimes_C B$ is injective, $0 \to A \to B \overset{\sigma^* - p_2^*}{\to} R \otimes_C B$ is also exact. This implies that $B^G = A$ and that $\pi_*(\mathcal{O}_X)^G = \mathcal{O}_Y$. Hence we see that $\pi : X \to Y$ is a geometric quotient by G. For all morphisms $Y' \to Y$, $\pi' : X' := X \times_Y Y' \to Y'$ is also a geometric quotient by G because π' and $G \times_S X' \to X' \times_{Y'} X'$ are faithfully flat. This completes the proof. $\square$

Lemma 4.15. *Let X and Y be schemes over a scheme S. Let G be a group scheme over S. Assume that an S-morphism $\sigma : G \times_S X \to X$ is a group action of G on X and that the action of G on Y is trivial. Let $\pi : X \to Y$ be a G-equivariant S-morphism and $s : Y \to X$ be an S-morphism such that $\pi \circ s = 1_Y$. Suppose that*

(i) *π is faithfully flat and locally of finite presentation, and*

(ii) *$G \times_S Y \overset{(1_G, s)}{\longrightarrow} G \times_S X \overset{\sigma}{\to} X$ is faithfully flat.*

Then $\pi : X \to Y$ is a universal geometric quotient by G.

Proof. From the assumption (i), $\pi : X \to Y$ is surjective and universally open. Let us show that the image of the morphism $(\sigma, p_2) : G \times_S X \to X \times_S X$ is $X \times_Y X$. Assume that $(f_1, f_2) \in X(\Omega) \times_{Y(\Omega)} X(\Omega)$ is a Ω-valued point with a field Ω. Set $h := \pi \circ f_1 = \pi \circ f_2 : \mathrm{Spec}\, \Omega \to Y$ and $f := s \circ h : \mathrm{Spec}\, \Omega \to X$. By the condition (ii), there exist $(g_i, f) : \mathrm{Spec}\, \Omega \to G \times_S X$ such that $\sigma \circ (g_i, f) = f_i$ for each $i = 1, 2$ (if necessary, take an extension field of Ω). Then $(\sigma, p_2) \circ (g_1 g_2^{-1}, f_2) : \mathrm{Spec}\, \Omega \to G \times_S X \to X \times_S X$ is (f_1, f_2). Hence the image of (σ, p_2) is $X \times_Y X$.

Let us show that $\pi_*(\mathcal{O}_X)^G = \mathcal{O}_Y$. Since π is faithfully flat, $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is injective. For an open set U of Y, put $V := \pi^{-1}(U)$. Assume that $\phi : V \to \mathbb{A}^1_S$ satisfies $\phi \circ (\sigma|_{G \times_S V}) = \phi \circ (p_2|_{G \times_S V})$. Set $\psi := \phi \circ (s|_U) : U \to \mathbb{A}^1_S$. By the assumption (ii), $G \times_S U \overset{(1_G, s|_U)}{\longrightarrow} G \times_S V \overset{\sigma|_{G \times_S V}}{\to} V$ is faithfully flat. Put $\Phi := (\sigma\, |_{G \times_S V}) \circ (1_G, s\, |_U)$. It is easy to verify that $\phi \circ \Phi = \psi \circ (\pi|_V) \circ \Phi$. Since $\mathcal{O}_V \to \Phi_*(\mathcal{O}_{G \times_S V})$ is injective, $\phi = \psi \circ (\pi|_V)$. This implies that $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is surjective, and hence $\pi_*(\mathcal{O}_X)^G = \mathcal{O}_Y$.

The assumptions (i) and (ii) are stable under any base change $Y' \to Y$. Therefore $\pi : X \to Y$ is a universal geometric quotient by G. $\square$

Remark 4.16. We can extend Lemma 4.15 in the following way: Let S, G, X, Y, and $\pi : X \to Y$ be as above. Let $Y = \cup_{i \in I} Y_i$ be an open covering of Y. Set $X_i := \pi^{-1}(Y_i)$. Let $s_i : Y_i \to X_i$ be an S-morphism with $\pi|_{X_i} \circ s_i = 1_{Y_i}$ for each $i \in I$. Suppose that (i) and the following (ii)' hold:

(ii)' $G \times_S Y_i \xrightarrow{(1_G, s_i)} G \times_S X_i \xrightarrow{\sigma} X_i$ is faithfully flat for each $i \in I$.

Then $\pi : X \to Y$ is a universal geometric quotient by G.

Theorem 4.17. *The morphism* $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ *is a universal geometric quotient by* PGL_2*.*

Proof. Obviously, π is locally of finite presentation. By Propositions 4.9 and 4.12, π and $\mathrm{Ch}_2(\Upsilon_1) \times \mathrm{PGL}_2 \xrightarrow{\sigma \circ (s, id)} \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}$ are faithfully flat. Hence Lemma 4.15 implies that $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ is a universal geometric quotient by PGL_2. (Of course, Lemma 4.15 also holds for right group actions.) $\square$

Corollary 4.18. *The prototype* $\pi_{\Upsilon_1, \mathrm{s.s.}} : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.}}$ *is a universal geometric quotient by* PGL_2*.*

Proof. The morphism $\pi_{\Upsilon_1, \mathrm{s.s.}} : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.}}$ is a base change of $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ by $\mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Upsilon_1)$. The statement follows from Theorem 4.17. $\square$

Corollary 4.19. *The prototype* $\pi_{\mathrm{F}_1, \mathrm{s.s.}} : \mathrm{Rep}_2(\mathrm{F}_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\mathrm{F}_1)_{\mathrm{s.s.}}$ *for group representations is a universal geometric quotient by* PGL_2*.*

Proof. The morphism $\pi_{\mathrm{F}_1, \mathrm{s.s.}} : \mathrm{Rep}_2(\mathrm{F}_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\mathrm{F}_1)_{\mathrm{s.s.}}$ is a base change of $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.}}$ by $\mathrm{Ch}_2(\mathrm{F}_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.}}$. The statement follows from the previous corollary. $\square$

Proposition 4.20. *Let* $\mathcal{A} \subseteq \mathrm{M}_2(R)$ *be a rank* 2 *semi-simple mold over a commutative ring* R*. Suppose that there exists* $A \in \mathcal{A}$ *such that* $m(A) \in R^\times$*. Then the following bilinear form is perfect:*

$$\begin{array}{rccc} \mathrm{tr}(\cdot\cdot) & : & \mathcal{A} \times \mathcal{A} & \to & R \\ & & (X, Y) & \mapsto & \mathrm{tr}(XY). \end{array}$$

In other words, the R*-linear map* $\mathcal{A} \to \mathrm{Hom}_R(\mathcal{A}, R)$ *defined by* $X \mapsto (Y \mapsto \mathrm{tr}(XY))$ *is an isomorphism. In particular, for each* $X \in \mathcal{A}$*, we* *have*

$$\begin{aligned} X &= (I_2, A)\begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(A) \\ \mathrm{tr}(A) & \mathrm{tr}(A^2) \end{pmatrix}^{-1}\begin{pmatrix} \mathrm{tr}(X) \\ \mathrm{tr}(AX) \end{pmatrix} \\ &= (I_2, A)\,\frac{1}{m(A)}\begin{pmatrix} \mathrm{tr}(A^2) & -\mathrm{tr}(A) \\ -\mathrm{tr}(A) & 2 \end{pmatrix}\begin{pmatrix} \mathrm{tr}(X) \\ \mathrm{tr}(AX) \end{pmatrix}. \end{aligned}$$

Proof. Remark that $\{I_2, A\}$ forms a basis of $\mathcal{A}$ over R. The determinant of the matrix

$$\begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(A) \\ \mathrm{tr}(A) & \mathrm{tr}(A^2) \end{pmatrix}$$

is equal to $m(A) \in R^{\times}$, and hence the inverse matrix exists.

For each $X = aI_2 + bA \in \mathcal{A}$ with $a, b \in R$, we have

$$\begin{pmatrix} \mathrm{tr}(X) \\ \mathrm{tr}(AX) \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(A) \\ \mathrm{tr}(A) & \mathrm{tr}(A^2) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Since

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(A) \\ \mathrm{tr}(A) & \mathrm{tr}(A^2) \end{pmatrix}^{-1}\begin{pmatrix} \mathrm{tr}(X) \\ \mathrm{tr}(AX) \end{pmatrix},$$

we have

$$\begin{aligned} X &= (I_2, A)\begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(A) \\ \mathrm{tr}(A) & \mathrm{tr}(A^2) \end{pmatrix}^{-1}\begin{pmatrix} \mathrm{tr}(X) \\ \mathrm{tr}(AX) \end{pmatrix} \\ &= (I_2, A)\,\frac{1}{m(A)}\begin{pmatrix} \mathrm{tr}(A^2) & -\mathrm{tr}(A) \\ -\mathrm{tr}(A) & 2 \end{pmatrix}\begin{pmatrix} \mathrm{tr}(X) \\ \mathrm{tr}(AX) \end{pmatrix}. \end{aligned}$$

Therefore we see that $\mathrm{tr}(\cdot\cdot) : \mathcal{A} \times \mathcal{A} \to R$ is perfect. $\square$

Proposition 4.21. *For each element γ of a monoid Γ, $(A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2})_{m(\sigma_\Gamma(\gamma))}$ is generated by $\{\mathrm{tr}(\sigma_\Gamma(\delta)) \mid \delta \in \Gamma\}$ and $m(\sigma_\Gamma(\gamma)))^{-1}$ over $\mathbb{Z}$.*

Proof. By the definition of $A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2}$, $(A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2})_{m(\sigma_\Gamma(\gamma))}$ is generated by $\{\mathrm{tr}(\sigma_\Gamma(\delta)), \det(\sigma_\Gamma(\delta)) \mid \delta \in \Gamma\}$ and $m(\sigma_\Gamma(\gamma)))^{-1}$ over $\mathbb{Z}$. Let S be the subalgebra of $(A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2})_{m(\sigma_\Gamma(\gamma))}$ generated by $\{\mathrm{tr}(\sigma_\Gamma(\delta)) \mid \delta \in \Gamma\}$ and $m(\sigma_\Gamma(\gamma)))^{-1}$ over $\mathbb{Z}$. It only suffices to prove that $\det(\sigma_\Gamma(\delta)) \in S$ for each $\delta \in \Gamma$. Using Proposition 4.20, we have

$$\sigma_\Gamma(\delta) = (\sigma_\Gamma(e), \sigma_\Gamma(\gamma))\begin{pmatrix} \mathrm{tr}(\sigma_\Gamma(e)) & \mathrm{tr}(\sigma_\Gamma(\gamma)) \\ \mathrm{tr}(\sigma_\Gamma(\gamma)) & \mathrm{tr}(\sigma_\Gamma(\gamma^2)) \end{pmatrix}^{-1}\begin{pmatrix} \mathrm{tr}(\sigma_\Gamma(\delta)) \\ \mathrm{tr}(\sigma_\Gamma(\gamma\delta)) \end{pmatrix}$$

in $\mathrm{M}_2((A_2(\Gamma)_{\mathrm{rk}\leq 2})_{m(\sigma_\Gamma(\gamma))})$ for each $\delta \in \Gamma$. Since the determinant of the matrix

$$T := \begin{pmatrix} \mathrm{tr}(\sigma_\Gamma(e)) & \mathrm{tr}(\sigma_\Gamma(\gamma)) \\ \mathrm{tr}(\sigma_\Gamma(\gamma)) & \mathrm{tr}(\sigma_\Gamma(\gamma^2)) \end{pmatrix}$$

is equal to $m(\sigma_\Gamma(\gamma)) = 2\mathrm{tr}(\sigma_\Gamma(\gamma^2)) - (\mathrm{tr}(\sigma_\Gamma(\gamma)))^2$, $\sigma_\Gamma(\delta) = aI_2 + b\sigma_\Gamma(\gamma)$ with some $a, b \in S$ for each $\delta \in \Gamma$. By Lemma 4.10, the statement follows from the claim that $\det(\sigma_\Gamma(\gamma)) \in S$. Let us prove the claim. Putting $\delta = \gamma^2$, we have

$$\begin{aligned} \sigma_\Gamma(\gamma^2) &= (\sigma_\Gamma(e), \sigma_\Gamma(\gamma))\, T^{-1} \begin{pmatrix} \mathrm{tr}(\sigma_\Gamma(\gamma^2)) \\ \mathrm{tr}(\sigma_\Gamma(\gamma^3)) \end{pmatrix} \\ &= (I_2, \sigma_\Gamma(\gamma)) \frac{1}{m(\sigma_\Gamma(\gamma))} \begin{pmatrix} \mathrm{tr}(\sigma_\Gamma(\gamma^2))^2 - \mathrm{tr}(\sigma_\Gamma(\gamma))\mathrm{tr}(\sigma_\Gamma(\gamma^3)) \\ -\mathrm{tr}(\sigma_\Gamma(\gamma))\mathrm{tr}(\sigma_\Gamma(\gamma^2)) + 2\mathrm{tr}(\sigma_\Gamma(\gamma^3)) \end{pmatrix}. \end{aligned}$$

We also obtain $\sigma_\Gamma(\gamma^2) = \mathrm{tr}(\sigma_\Gamma(\gamma))\sigma_\Gamma(\gamma) - \det \sigma_\Gamma(\gamma) I_2$ by the Cayley-Hamilton Theorem. Comparing the coefficients of I_2, we have

$$\det(\sigma_\Gamma(\gamma)) = \frac{\mathrm{tr}(\sigma_\Gamma(\gamma))\mathrm{tr}(\sigma_\Gamma(\gamma^3)) - \mathrm{tr}(\sigma_\Gamma(\gamma^2))^2}{m(\sigma_\Gamma(\gamma))}.$$

Hence we have proved the statement. □

Let Γ_1, Γ_2 be monoids. Let $\psi : \Gamma_1 \to \Gamma_2$ be a monoid homomorphism. Then ψ induces canonical ring homomorphisms $\psi_* : A_2(\Gamma_1)_{\mathrm{rk}\leq 2} \to A_2(\Gamma_2)_{\mathrm{rk}\leq 2}$ and $\psi_* : A_2(\Gamma_1)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2} \to A_2(\Gamma_2)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2}$. Set $\gamma_2 := \psi(\gamma_1)$ for $\gamma_1 \in \Gamma_1$. We obtain the ring homomorphisms

$$\begin{aligned} \psi_* : (A_2(\Gamma_1)_{\mathrm{rk}\leq 2})_{m(\sigma_{\Gamma_1}(\gamma_1))} &\to (A_2(\Gamma_2)_{\mathrm{rk}\leq 2})_{m(\sigma_{\Gamma_2}(\gamma_2))}, \\ \psi_* : (A_2(\Gamma_1)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2})_{m(\sigma_{\Gamma_1}(\gamma_1))} &\to (A_2(\Gamma_2)^{\mathrm{Ch}}_{\mathrm{rk}\leq 2})_{m(\sigma_{\Gamma_2}(\gamma_2))}. \end{aligned}$$

Hence we have the morphisms

$$\begin{array}{ccc} \mathrm{Rep}_2(\Gamma_2)_{\mathrm{rk}\leq 2} & \stackrel{\psi^*}{\to} & \mathrm{Rep}_2(\Gamma_1)_{\mathrm{rk}\leq 2} \\ \downarrow & & \downarrow \\ \mathrm{Ch}_2(\Gamma_2)_{\mathrm{rk}\leq 2} & \stackrel{\psi^*}{\to} & \mathrm{Ch}_2(\Gamma_1)_{\mathrm{rk}\leq 2} \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Rep}_2(\Gamma_2)_{\mathrm{s.s.},\gamma_2} & \stackrel{\psi^*}{\to} & \mathrm{Rep}_2(\Gamma_1)_{\mathrm{s.s.},\gamma_1} \\ \downarrow & & \downarrow \\ \mathrm{Ch}_2(\Gamma_2)_{\mathrm{s.s.},\gamma_2} & \stackrel{\psi^*}{\to} & \mathrm{Ch}_2(\Gamma_1)_{\mathrm{s.s.},\gamma_1}. \end{array}$$

Lemma 4.22. *Let γ be an element of a monoid Γ. Let $\psi : \Upsilon_1 \to \Gamma$ be the monoid homomorphism sending α to γ. Then ψ induces the following diagram which is a fibre product:*

$$\begin{array}{ccc} \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma} & \stackrel{\pi_{\Gamma,\mathrm{s.s.},\gamma}}{\to} & \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma} \\ \downarrow & & \downarrow \\ \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.},\alpha} & \to & \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.},\alpha}. \end{array}$$

In particular, the morphism $\pi_{\Gamma,\mathrm{s.s.},\gamma}$ can be obtained by base change of the prototype for each $\gamma \in \Gamma$.

Proof. Put $X := \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma}$ and $Y := \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.},\alpha} \times_{\mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.},\alpha}} \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}$. We shall show that the morphism $f : X \to Y$ induced by ψ is an isomorphism. Let R be a commutative ring. For two R-valued points σ_1, σ_2 of X, assume that $f(\sigma_1) = f(\sigma_2)$ as R-valued points of Y. Considering σ_1 and σ_2 as representations of degree 2 for Γ in R, we have

$$\sigma_i(\delta) = (I_2, \sigma_i(\gamma)) \begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(\sigma_i(\gamma)) \\ \mathrm{tr}(\sigma_i(\gamma)) & \mathrm{tr}(\sigma_i(\gamma^2)) \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{tr}(\sigma_i(\delta)) \\ \mathrm{tr}(\sigma_i(\delta\gamma)) \end{pmatrix}$$

for each $\delta \in \Gamma$ and $i = 1, 2$. Since $f(\sigma_1) = f(\sigma_2)$, we obtain $\sigma_1(\gamma) = f(\sigma_1)(\alpha) = f(\sigma_2)(\alpha) = \sigma_2(\gamma)$ and $\mathrm{tr}(\sigma_1(\delta')) = \mathrm{tr}(\sigma_2(\delta'))$ for each $\delta' \in \Gamma$. Hence we have $\sigma_1 = \sigma_2$.

Let $y = (\rho, \chi)$ be an R-valued point of Y. Here $\rho : \Upsilon_1 \to \mathrm{M}_2(R)$ and χ are R-valued points of $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.},\alpha}$ and $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}$, respectively. Let us denote by $\chi(\delta)$ the image of $\mathrm{tr}(\sigma_\Gamma(\delta))$ by the ring homomorphism $\phi : (A_2(\Gamma)^{\mathrm{Ch}}_{\mathrm{rk}\le 2})_{m(\sigma_\Gamma(\gamma))} \to R$ associated to χ. Then $\phi(m(\sigma_\Gamma(\gamma))) = m(\rho(\alpha))$ and $\chi(\gamma^m) = \mathrm{tr}(\rho(\alpha^m))$ for $m \in \mathbb{N}$. We define the map $\sigma : \Gamma \to \mathrm{M}_2(R)$ by

$$\sigma(\delta) := (I_2, \rho(\alpha)) \begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(\rho(\alpha)) \\ \mathrm{tr}(\rho(\alpha)) & \mathrm{tr}(\rho(\alpha^2)) \end{pmatrix}^{-1} \begin{pmatrix} \chi(\delta) \\ \chi(\delta\gamma) \end{pmatrix}$$

for $\delta \in \Gamma$. It is easy to see that $\sigma(e) = I_2$ and $\sigma(\gamma) = \rho(\alpha)$.

Note that

$$\sigma_\Gamma(\delta) = (I_2, \sigma_\Gamma(\gamma)) \begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(\sigma_\Gamma(\gamma)) \\ \mathrm{tr}(\sigma_\Gamma(\gamma)) & \mathrm{tr}(\sigma_\Gamma(\gamma^2)) \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{tr}(\sigma_\Gamma(\delta)) \\ \mathrm{tr}(\sigma_\Gamma(\gamma\delta)) \end{pmatrix}$$

in $\mathrm{M}_2((A_2(\Gamma)_{\mathrm{rk}\le 2})_{m(\sigma_\Gamma(\gamma))})$ for each $\delta \in \Gamma$. By a similar discussion in the proof of [10, Theorem 5.1], we see that σ is a representation, and hence that σ can be regarded as an R-valued point of X. Since $\mathrm{tr}(\sigma(\delta)) = \chi(\delta)$ for each $\delta \in \Gamma$, we also see that $f(\sigma) = y$ by using Proposition 4.21. Therefore f is an isomorphism. $\square$

Remark 4.23. We can also prove the group version of Lemma 4.22: Let γ be an element of a group Γ. Let $\phi : \mathrm{F}_1 \to \Gamma$ be the group homomorphism sending α to γ. Then ϕ induces the following diagram which is a fibre product:

$$\begin{array}{ccc} \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma} & \to & \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma} \\ \downarrow & & \downarrow \\ \mathrm{Rep}_2(\mathrm{F}_1)_{\mathrm{s.s.},\alpha} & \to & \mathrm{Ch}_2(\mathrm{F}_1)_{\mathrm{s.s.},\alpha}. \end{array}$$

In particular, the morphism $\pi_{\Gamma,\mathrm{s.s.},\gamma}$ can be obtained by base change of the prototype for group representations.

Corollary 4.24. *The morphism* $\pi_{\Gamma,\mathrm{s.s.}} : \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ *is a universal geometric quotient by* PGL_2 *for a group or a monoid* Γ.

Proof. From Lemma 4.22 and Corollary 4.18 we see that $\pi_{\Gamma,\mathrm{s.s.}}$ gives a universal geometric quotient. □

Remark 4.25. The morphism $\pi_{\Gamma,\mathrm{s.s.}} : \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ is smooth and surjective. Indeed, the prototype $\pi_{\Upsilon_1,\mathrm{s.s.}} : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.}}$ is smooth and surjective because it is obtained by base change of $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \to \mathrm{Ch}_2(\Upsilon_1)$ and π is smooth and surjective by Proposition 4.9.

Remark 4.26. For each point $x \in \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$, there exists a local section $s_x : V_x \to \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}}$ on a neighbourhood V_x of x such that $\pi_{\Gamma,\mathrm{s.s.}} \circ s_x = id_{V_x}$. Indeed, take $\gamma \in \Gamma$ such that $x \in \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}$. By Lemma 4.22, $\pi_{\Gamma,\mathrm{s.s.}} : \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma} \to \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}$ has a section $s_{\Gamma,\gamma}$ because $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{s.s.}} \to \mathrm{Ch}_2(\Upsilon_1)_{\mathrm{s.s.}}$ has a section s. Hence we can take $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}$ as a neighbourhood V_x of x.

Lemma 4.27. *Let* ρ_1, ρ_2 *be representations with semi-simple mold for a group (or a monoid)* Γ *on a scheme* X. *Let* $f_i : X \to \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}}$ *be the morphism associated to* ρ_i *for* $i = 1, 2$. *If* $\pi_{\Gamma,\mathrm{s.s.}} \circ f_1 = \pi_{\Gamma,\mathrm{s.s.}} \circ f_2 : X \to \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$, *then for each* $x \in X$ *there exists* $P_x \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ *on a neighbourhood* V_x *of* x *such that* $P_x^{-1}\rho_1 P_x = \rho_2$ *on* V_x.

Proof. For $x \in X$, take $\gamma \in \Gamma$ such that $(\pi_{\Gamma,\mathrm{s.s.}} \circ f_1)(x) = (\pi_{\Gamma,\mathrm{s.s.}} \circ f_2)(x) \in \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}$. We may assume that $f_i : X \to \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma}$ for $i = 1, 2$ from the beginning. By Remark 4.26, $\pi_{\Gamma,\mathrm{s.s.}} : \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma} \to \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma}$ has a section $s_{\Gamma,\gamma}$. Let ρ_3 be the representations with semi-simple mold on X associated to $s_{\Gamma,\gamma} \circ \pi_{\Gamma,\mathrm{s.s.}} \circ f_1 = s_{\Gamma,\gamma} \circ \pi_{\Gamma,\mathrm{s.s.}} \circ f_2$. Note that

$$\rho_3(\gamma) = \begin{pmatrix} 0 & -\det(\rho_1(\gamma)) \\ 1 & \mathrm{tr}(\rho_1(\gamma)) \end{pmatrix} = \begin{pmatrix} 0 & -\det(\rho_2(\gamma)) \\ 1 & \mathrm{tr}(\rho_2(\gamma)) \end{pmatrix}$$

and that $\mathrm{tr}(\rho_1(\delta)) = \mathrm{tr}(\rho_2(\delta)) = \mathrm{tr}(\rho_3(\delta))$ for each $\delta \in \Gamma$. There exist $Q_1, Q_2 \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood V_x of x such that $Q_1^{-1}\rho_1(\gamma)Q_1 = \rho_3(\gamma)$ and $Q_2^{-1}\rho_2(\gamma)Q_2 = \rho_3(\gamma)$ by Lemma 4.8. Since

$$\rho_i(\delta) = (I_2, \rho_i(\gamma)) \begin{pmatrix} \mathrm{tr}(I_2) & \mathrm{tr}(\rho_i(\gamma)) \\ \mathrm{tr}(\rho_i(\gamma)) & \mathrm{tr}(\rho_i(\gamma^2)) \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{tr}(\rho_i(\delta)) \\ \mathrm{tr}(\rho_i(\gamma\delta)) \end{pmatrix}$$

on V_x for $\delta \in \Gamma$ and for $i = 1, 2, 3$, we have $Q_1^{-1}\rho_1(\delta)Q_1 = \rho_3(\delta)$ and $Q_2^{-1}\rho_2(\delta)Q_2 = \rho_3(\delta)$ for each $\delta \in \Gamma$. Hence $(Q_1Q_2^{-1})^{-1}\rho_1(Q_1Q_2^{-1}) = \rho_2$ on V_x. This completes the proof. □

Theorem 4.28. *Let R be a local ring. For two representations with semi-simple mold $\rho_1, \rho_2 : \Gamma \to \mathrm{GL}_2(R)$ for a group (or a monoid) Γ, ρ_1 and ρ_2 are equivalent to each other (in other words, there exists $P \in \mathrm{GL}_2(R)$ such that $P^{-1}\rho_1(\gamma)P = \rho_2(\gamma)$ for any $\gamma \in \Gamma$) if and only if* $\mathrm{tr}(\rho_1(\gamma)) = \mathrm{tr}(\rho_2(\gamma))$ *for each* $\gamma \in \Gamma$.

Proof. Let f_1, f_2 be the R-valued points of $\mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.}}$ associated to ρ_1, ρ_2, respectively. Using Proposition 4.21 and $m(\rho_i(\gamma)) = 2\mathrm{tr}(\rho_i(\gamma^2)) - (\mathrm{tr}(\rho_i(\gamma)))^2$ for $i = 1, 2$, we see that $\mathrm{tr}(\rho_1(\gamma)) = \mathrm{tr}(\rho_2(\gamma))$ for each $\gamma \in \Gamma$ if and only if $\pi_{\Gamma,\mathrm{s.s.}} \circ f_1 = \pi_{\Gamma,\mathrm{s.s.}} \circ f_2$ as R-valued points of $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$. The statement follows from Lemma 4.27. $\square$

Let us define $\mathcal{E}q\mathcal{SS}_2(\Gamma)$ as the sheafification of the following contravariant functor with respect to Zariski topology:

$$\begin{array}{ccl} (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \{\rho \mid \text{ rep. with s.s. mold for } \Gamma \text{ on } X\}/\sim . \end{array}$$

By a *generalized representation with semi-simple mold* for Γ on a scheme X, we understand pairs $\{(U_i, \rho_i)\}_{i\in I}$ of an open set U_i and a representation with semi-simple mold $\rho_i : \Gamma \to \mathrm{M}_2(\Gamma(U_i, \mathcal{O}_X))$ satisfying the following two conditions:

(i) $\cup_{i\in I} U_i = X$,
(ii) for each $x \in U_i \cap U_j$, there exists $P_x \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood $V_x \subseteq U_i \cap U_j$ of x such that $P_x^{-1}\rho_i P_x = \rho_j$ on V_x.

Generalized representations with semi-simple mold $\{(U_i, \rho_i)\}_{i\in I}$ and $\{(V_j, \sigma_j)\}_{j\in J}$ are called *equivalent* if $\{(U_i, \rho_i)\}_{i\in I} \cup \{(V_j, \sigma_j)\}_{j\in J}$ is a generalized representation with semi-simple mold again. We easily see that $\mathcal{E}q\mathcal{SS}_2(\Gamma)(X)$ is the set of equivalence classes of generalized representations with semi-simple mold for Γ on a scheme X.

Theorem 4.29. *The scheme* $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ *is a fine moduli scheme associated to the functor* $\mathcal{E}q\mathcal{SS}_2(\Gamma)$ *for a group or a monoid* Γ*:*

$$\begin{array}{cccl} \mathcal{E}q\mathcal{SS}_2(\Gamma) & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{r} \textit{gen. rep. with s.s. mold} \\ \textit{for } \Gamma \textit{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

In other words, $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ *represents the functor* $\mathcal{E}q\mathcal{SS}_2(\Gamma)$*. The moduli* $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ *is separated over* $\mathbb{Z}$*; if* Γ *is a finitely generated group or monoid, then* $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ *is of finite type over* $\mathbb{Z}$*.*

Proof. It is easy to define a canonical morphism $\mathcal{E}q\mathcal{SS}_2(\Gamma) \to h_{\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}} := \mathrm{Hom}(-, \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}})$. Let us define a morphism $h_{\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}} \to \mathcal{E}q\mathcal{SS}_2(\Gamma)$. Let $g \in h_{\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}}(X)$ with a scheme X. For each $x \in X$, take $\gamma_x \in \Gamma$ such that $g(x) \in \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma_x}$. By using the section $s_{\Gamma,\gamma_x} : \mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.},\gamma_x} \to \mathrm{Rep}_2(\Gamma)_{\mathrm{s.s.},\gamma_x}$ in Remark 4.26, we can define a representation with semi-simple mold ρ_x on a neighbourhood U_x of x. By Lemma 4.27, we see that $\{(U_x, \rho_x)\}_{x\in X} \in \mathcal{E}q\mathcal{SS}_2(\Gamma)(X)$ and that the morphism $h_{\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}} \to \mathcal{E}q\mathcal{SS}_2(\Gamma)$ is well-defined. It is easy to see that $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ represents the functor $\mathcal{E}q\mathcal{SS}_2(\Gamma)$.

Since $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ is an open subscheme of the affine scheme $\mathrm{Ch}_2(\Gamma)_{\mathrm{rk}\leq 2}$, $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ is separated over $\mathbb{Z}$. Suppose that Γ is finitely generated. Then $\mathrm{Ch}_2(\Gamma)_{\mathrm{rk}\leq 2}$ is of finite type over $\mathbb{Z}$ by Proposition 4.4. Hence $\mathrm{Ch}_2(\Gamma)_{\mathrm{s.s.}}$ is also of finite type over $\mathbb{Z}$. □

Remark 4.30. Let A be an associative algebra over a commutative ring R. For an R-scheme X, we say that an R-algebra homomorphism $\rho : A \to \mathrm{M}_2(\Gamma(X, \mathcal{O}_X))$ is a 2-*dimensional representation* of A on X. For a 2-dimensional representation ρ of A, ρ is called a *representation with semi-simple mold* if the subalgebra $\rho(A)$ of $\mathrm{M}_2(\mathcal{O}_X)$ generates a semi-simple mold on X. In a similar way as group or monoid cases, we can define generalized representations with semi-simple mold for A on an R-scheme X. The contravariant functor $\mathcal{E}q\mathcal{SS}_2(A)$ from the category of R-schemes to the category of sets is defined as

$$\begin{array}{rccl} \mathcal{E}q\mathcal{SS}_2(A) & : & (\mathbf{Sch}/R)^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{r} \text{gen. rep. with s.s. mold} \\ \text{for } A \text{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

Then we can construct the fine moduli $\mathrm{Ch}_2(A)_{\mathrm{s.s.}}$ associated to $\mathcal{E}q\mathcal{SS}_2(A)$ in the same way as Theorem 4.29. The moduli $\mathrm{Ch}_2(A)_{\mathrm{s.s.}}$ is separated over R. If A is a finitely generated algebra over R, then $\mathrm{Ch}_2(A)_{\mathrm{s.s.}}$ is of finite type over R. For a local ring S over R, we see that two representations with semi-simple mold $\rho_1, \rho_2 : A \to \mathrm{M}_2(S)$ are equivalent to each other (in other words, there exists $P \in \mathrm{GL}_2(S)$ such that $P^{-1}\rho_1(a)P = \rho_2(a)$ for any $a \in A$) if and only if $\mathrm{tr}(\rho_1(a)) = \mathrm{tr}(\rho_2(a))$ for each $a \in A$ (the associative algebra version of Theorem 4.28).

Remark 4.31. We have introduced the notion of generalized representations with semi-simple mold for describing the moduli functors $\mathcal{E}q\mathcal{SS}_2(\Gamma)$ and $\mathcal{E}q\mathcal{SS}_2(A)$. However, the moduli functors can also be described as $\mathcal{E}q\mathcal{SS}'_2(\Gamma)$ and $\mathcal{E}q\mathcal{SS}'_2(A)$ by using the notion of representations generating sheaves of algebras which define semi-simple molds. More precisely, see §8.

5. Unipotent mold ($ch \neq 2$ case)

Recall that a rank 2 mold $\mathcal{A} \subseteq \mathrm{M}_2(\mathcal{O}_X)$ over a $\mathbb{Z}[1/2]$-scheme X is called *unipotent* if $m(s) := \mathrm{tr}(s)^2 - 4\det(s) = 0$ for each open subset

$U \subseteq X$ and for each $s \in \mathcal{A}(U)$. In this section, all schemes are over $\operatorname{Spec}\mathbb{Z}[1/2]$ and all commutative rings are over $\mathbb{Z}[1/2]$. We construct the moduli of representations with unipotent mold over $\operatorname{Spec}\mathbb{Z}[1/2]$.

As seen in Theorem 4.17, $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ is a universal geometric quotient by PGL_2. Put $A := \sigma_{\Upsilon_1}(\alpha)$ for the universal representation σ_{Υ_1} of $\Upsilon_1 = \langle \alpha \rangle$. Let Z be the closed subscheme of $\mathrm{Ch}_2(\Upsilon_1)$ defined by $m(A) = \mathrm{tr}(A)^2 - 4\det(A) = 0$. By base change, we obtain a universal geometric quotient $\pi' : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \times_{\mathrm{Ch}_2(\Upsilon_1)} Z \to Z$ by PGL_2. However, this quotient π' is not so good, because Z has a singular fibre over $\mathbb{F}_2$ which is defined by $\mathrm{tr}(A)^2 = 0$. Therefore we assume that all schemes are over $\operatorname{Spec}\mathbb{Z}[1/2]$ in this section. The case of unipotent molds over $\mathbb{F}_2$ will be discussed in the next section.

Assume that R is a $\mathbb{Z}[1/2]$-algebra and that $A \subseteq \mathrm{M}_2(R)$ is a unipotent mold over R through this section.

Notation 5.1. For $X \in \mathrm{M}_2(R)$, we define $\eta(X) := X - \dfrac{\mathrm{tr}(X)}{2} I_2$.

Lemma 5.2. *Suppose that $X \in \mathrm{M}_2(R)$ satisfies $m(X) = \mathrm{tr}(X)^2 - 4\det(X) = 0$. Then $\eta(X)^2 = 0$.*

Proof. By the Cayley-Hamilton theorem, we have

$$\begin{aligned}
\eta(X)^2 &= X^2 - \mathrm{tr}(X)X + \frac{(\mathrm{tr}(X))^2}{4} I_2 \\
&= \mathrm{tr}(X)X - \det(X)I_2 - \mathrm{tr}(X)X + \frac{(\mathrm{tr}(X))^2}{4} I_2 \\
&= 0,
\end{aligned}$$

since $\mathrm{tr}(X)^2 = 4\det(X)$. □

Lemma 5.3. *Let R be a $\mathbb{Z}[1/2]$-algebra. Let $A \subseteq \mathrm{M}_2(R)$ be a unipotent mold over R. If $X, Y \in A$, then $2\mathrm{tr}(XY) = \mathrm{tr}(X)\mathrm{tr}(Y)$.*

Proof. Since we have only to prove that the equality holds locally, we may assume that there exists $Z \in A$ such that $A = R\cdot I_2 + R \cdot Z$ and $m(Z) = \mathrm{tr}(Z)^2 - 4\det(Z) = 0$. Put $X = aI_2 + bZ$ and $Y = cI_2 + dZ$. Then we have

$$\begin{aligned}
2\mathrm{tr}(XY) &= 2ac\,\mathrm{tr}(I_2) + 2(ad+bc)\,\mathrm{tr}(Z) + 2bd\,\mathrm{tr}(Z^2) \\
&= 4ac + 2(ad+bc)\,\mathrm{tr}(Z) + 2bd\,(\mathrm{tr}(Z))^2 - 4bd\,\det(Z) \\
&= 4ac + 2(ad+bc)\,\mathrm{tr}(Z) + 4bd\,\det(Z)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}(X)\mathrm{tr}(Y) &= (2a + b\,\mathrm{tr}(Z))(2c + d\,\mathrm{tr}(Z)) \\
&= 4ac + 2(ad + bc)\,\mathrm{tr}(Z) + bd\,(\mathrm{tr}(Z))^2 \\
&= 4ac + 2(ad + bc)\,\mathrm{tr}(Z) + 4bd\,\det(Z).
\end{aligned}$$

This completes the proof. $\square$

Notation 5.4. Let R and A be as above. For $X \in \mathrm{M}_2(R)$, we denote $\mathrm{tr}(X)/2$ by $r(X)$. Note that $\eta(X) = X - r(X)I_2$. From Lemma 5.3, we have $r(XY) = r(X)r(Y)$ for $X, Y \in A$. If $X = aI_2 + bZ$, then $\eta(X) = aI_2 + bZ - r(aI_2 + bZ)I_2 = b(Z - r(Z)I_2) = b\eta(Z)$.

Lemma 5.5. *Let R and A be as in Lemma 5.3. For $X, Y \in A$,*

$$\eta(XY) \;=\; r(X)\eta(Y) + \eta(X)r(Y).$$

Proof. As in the proof of Lemma 5.3, we may assume that there exists $Z \in A$ such that $A = R \cdot I_2 + R \cdot Z$ and $m(Z) = \mathrm{tr}(Z)^2 - 4\det(Z) = 0$. For $X, Y \in A$, there exists $\lambda, \mu \in R$ such that $\eta(X) = \lambda\eta(Z)$ and $\eta(Y) = \mu\eta(Z)$. Since $\eta(X)\eta(Y) = \lambda\mu\eta(Z)^2 = 0$, we have

$$\begin{aligned}
\eta(XY) &= XY - r(XY)I_2 \\
&= (X - r(X)I_2)(Y - r(Y)I_2) + r(Y)(X - r(X)I_2) \\
&\qquad + r(X)(Y - r(Y)I_2) \\
&= \eta(X)\eta(Y) + r(X)\eta(Y) + \eta(X)r(Y) \\
&= r(X)\eta(Y) + \eta(X)r(Y).
\end{aligned}$$

This completes the proof. $\square$

Notation 5.6. Let Γ be a group or a monoid. Let $\rho : \Gamma \to \mathrm{M}_2(R)$ be a representation with the unipotent mold A. For each $\gamma \in \Gamma$, we denote $\eta(\rho(\gamma))$ and $r(\rho(\gamma))$ by $\eta(\gamma)$ and $r(\gamma)$, respectively. Assume that there exists $\alpha \in \Gamma$ such that $A = R \cdot I_2 + R \cdot \rho(\alpha)$. Then $A = R \cdot I_2 + R \cdot \eta(\alpha)$ and for each $\gamma \in \Gamma$ there exists a unique $a_\alpha(\gamma) \in R$ such that $\eta(\gamma) = a_\alpha(\gamma)\eta(\alpha)$.

Remark 5.7. Note that the map $r(\cdot) : \Gamma \to R$ is a character of Γ. In other words, $r(e) = 1$ and $r(\gamma\delta) = r(\gamma)r(\delta)$ for $\gamma, \delta \in \Gamma$. From Lemma 5.5 we see that the map $a_\alpha(\cdot) : \Gamma \to R$ is a derivation with respect to r, that is, a_α satisfies the condition $a_\alpha(\gamma\delta) = r(\gamma)a_\alpha(\delta) + a_\alpha(\gamma)r(\delta)$ for each $\gamma, \delta \in \Gamma$.

For a representation $\rho : \Gamma \to \mathrm{M}_2(R)$ with the unipotent mold A such that $A = R \cdot I_2 + R \cdot \rho(\alpha)$ for some $\alpha \in \Gamma$, we have a character $r : \Gamma \to R$

and a derivation $a_\alpha : \Gamma \to R$ with respect to r. Conversely, a character and a derivation give us a representation with unipotent mold.

Lemma 5.8. *Let $r : \Gamma \to R$ be a character and let $a : \Gamma \to R$ be a derivation with respect to r. Assume that there exists $\alpha \in \Gamma$ such that $a(\alpha) \in R^\times$. Furthermore assume that there exists $Z \in \mathrm{M}_2(R)$ such that $A := R \cdot I_2 + R \cdot Z \subseteq \mathrm{M}_2(R)$ is a unipotent mold. Then the map*

$$\begin{array}{rcccl} \rho & : & \Gamma & \to & \mathrm{M}_2(R) \\ & & \gamma & \to & r(\gamma)I_2 + a(\gamma)\eta(Z) \end{array}$$

is a representation for Γ with the unipotent mold A.

Proof. For $\gamma, \delta \in \Gamma$, we have

$$\begin{aligned} \rho(\gamma)\rho(\delta) &= (r(\gamma)I_2 + a(\gamma)\eta(Z))(r(\delta)I_2 + a(\delta)\eta(Z)) \\ &= r(\gamma)r(\delta)I_2 + a(\gamma)r(\delta)\eta(Z) + r(\gamma)a(\delta)\eta(Z) \\ &= r(\gamma\delta)I_2 + a(\gamma\delta)\eta(Z) \\ &= \rho(\gamma\delta). \end{aligned}$$

Since $\rho(e) = I_2$ and $\rho(\alpha) = (r(\alpha) - a(\alpha)\mathrm{tr}(Z)/2)I_2 + a(\alpha)Z$, the map ρ is a representation with the unipotent mold A. □

Definition 5.9. Let us denote $\mathrm{Rep}_2(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ by $\mathrm{Rep}_2(\Gamma)[1/2]$. We define the subscheme $\mathrm{Rep}_2(\Gamma)_u$ of $\mathrm{Rep}_2(\Gamma)[1/2]$ by

$$\mathrm{Rep}_2(\Gamma)_u := \{\rho \in \mathrm{Rep}_2(\Gamma)[1/2] \mid \rho \text{ has a unipotent mold } \}.$$

We call $\mathrm{Rep}_2(\Gamma)_u$ the *unipotent part* of the representation variety of degree 2 for Γ over $\mathbb{Z}[1/2]$. Recall that $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\leq 2}$ is a closed subscheme of $\mathrm{Rep}_2(\Gamma)$ and that $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}2}$ is an open subscheme of $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\leq 2}$ (Definition 4.1). Set $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\leq 2}[1/2] := \mathrm{Rep}_2(\Gamma)_{\mathrm{rk}\leq 2} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ and $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}2}[1/2] := \mathrm{Rep}_2(\Gamma)_{\mathrm{rk}2} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$. Then $\mathrm{Rep}_2(\Gamma)_u$ is a closed subscheme of $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk}2}[1/2]$ defined by $m(\sigma_\Gamma(\gamma)) = 0$ for all $\gamma \in \Gamma$.

Definition 5.10. Let us $\mathrm{Rep}_1(\Gamma)[1/2]$ denote the representation variety $\mathrm{Rep}_1(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ of degree 1 for Γ over $\mathbb{Z}[1/2]$. Let us denote by $A_1(\Gamma)[1/2]$ the coordinate ring of $\mathrm{Rep}_1(\Gamma)[1/2]$. For an $A_1(\Gamma)[1/2]$-module M, we define the $A_1(\Gamma)[1/2]$-*module of derivations* by

$$\mathrm{Der}(\Gamma, M) := \left\{ a : \Gamma \to M \,\middle|\, \begin{array}{c} a(\gamma\delta) = \chi_\Gamma(\gamma)a(\delta) + a(\gamma)\chi_\Gamma(\delta) \\ \text{for each } \gamma, \delta \in \Gamma \end{array} \right\}.$$

Here we denote by $\chi_\Gamma : \Gamma \to A_1(\Gamma)[1/2]$ the universal representation of degree 1 for Γ.

Lemma 5.11. *There exists a universal $A_1(\Gamma)[1/2]$-module Ω_Γ representing the covariant functor*

$$\begin{array}{rcccc} \mathrm{Der}(\Gamma, -) & : & (A_1(\Gamma)[1/2]\text{-}\mathbf{Mod}) & \to & (A_1(\Gamma)[1/2]\text{-}\mathbf{Mod}) \\ & & M & \mapsto & \mathrm{Der}(\Gamma, M). \end{array}$$

In particular,

$$\mathrm{Der}(\Gamma, M) \xrightarrow{\sim} \mathrm{Hom}_{A_1(\Gamma)[1/2]}(\Omega_\Gamma, M)$$

is an isomorphism for each $A_1(\Gamma)[1/2]$-module M.

Proof. We define the $A_1(\Gamma)[1/2]$-module Ω_Γ by

$$\Omega_\Gamma := (\oplus_{\gamma \in \Gamma} A_1(\Gamma)[1/2] \cdot d\gamma)/N,$$

where N is the $A_1(\Gamma)[1/2]$-submodule generated by $\{\chi_\Gamma(\gamma)d\delta + \chi_\Gamma(\delta)d\gamma - d(\gamma\delta) \mid \gamma, \delta \in \Gamma\}$ of the free $A_1(\Gamma)[1/2]$-module $\oplus_{\gamma\in\Gamma} A_1[1/2](\Gamma) \cdot d\gamma$. It is easy to check that Ω_Γ represents the above functor. □

Remark 5.12. If Γ is a finitely generated group or monoid, then $A_1(\Gamma)[1/2]$ is a finitely generated algebra over $\mathbb{Z}[1/2]$ and Ω_Γ is a finitely generated $A_1(\Gamma)[1/2]$-module. Indeed, let $S = \{\alpha_1, \cdots, \alpha_n\}$ be a set of generators of Γ. We may assume that α_i^{-1} is also an element of S for each $1 \le i \le n$ if Γ is a group. Then $A_1(\Gamma)[1/2]$ is generated by $\{\chi_\Gamma(\alpha_1), \ldots, \chi_\Gamma(\alpha_n)\}$ over $\mathbb{Z}[1/2]$ and Ω_Γ is generated by $\{d(\alpha_1), \ldots, d(\alpha_n)\}$ over $A_1(\Gamma)[1/2]$.

Definition 5.13. We define the scheme $\mathrm{Ch}_2(\Gamma)_u$ over $A_1(\Gamma)[1/2]$ by

$$\mathrm{Ch}_2(\Gamma)_u := \mathrm{Proj} S(\Omega_\Gamma),$$

where $S(\Omega_\Gamma)$ is the the symmetric algebra of Ω_Γ over $A_1(\Gamma)[1/2]$.

Example 5.14. Let $\Upsilon_1 = \langle \alpha_0 \rangle$ be the free monoid of rank 1. The $A_1(\Upsilon_1)[1/2]$-module Ω_{Υ_1} is isomorphic to $A_1(\Upsilon_1)[1/2]$. Indeed, the $A_1(\Upsilon_1)[1/2]$-module homomorphism

$$\begin{array}{ccc} A_1(\Upsilon_1)[1/2] & \to & \Omega_{\Upsilon_1} \\ 1 & \mapsto & d\alpha_0 \end{array}$$

gives an isomorphism. In particular, $\mathrm{Ch}_2(\Upsilon_1)_u \cong \mathrm{Rep}_1(\Upsilon_1)[1/2]$.

Let $\psi : X \to \mathrm{Rep}_1(\Gamma)[1/2]$ be a $\mathbb{Z}[1/2]$-morphism. Let us regard Ω_Γ as a quasi-coherent sheaf on $\mathrm{Rep}_1(\Gamma)[1/2]$. There exists a one-to-one correspondence

$$\mathrm{Hom}_{\,\mathrm{Rep}_1(\Gamma)[1/2]}\,(X, \mathrm{Ch}_2(\Gamma)_u) \cong \{\psi^*(\Omega_\Gamma) \twoheadrightarrow \mathcal{L} \to 0 \mid \mathcal{L} \text{ is a line bundle on } X\}/\sim .$$

Here we say that $\psi^*(\Omega_\Gamma) \overset{f_1}{\twoheadrightarrow} \mathcal{L}_1$ and $\psi^*(\Omega_\Gamma) \overset{f_2}{\twoheadrightarrow} \mathcal{L}_2$ are equivalent if there exists an isomorphism $g : \mathcal{L}_1 \overset{\cong}{\to} \mathcal{L}_2$ such that $g \circ f_1 = f_2$.

The group scheme $\mathrm{PGL}_2[1/2] := \mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ over $\mathbb{Z}[1/2]$ acts on $\mathrm{Rep}_2(\Gamma)_u$ by $\rho \mapsto P^{-1}\rho P$. We define the morphism $\lambda : \mathrm{Rep}_2(\Gamma)_u \to \mathrm{Rep}_1(\Gamma)[1/2]$ by $\rho \mapsto r = \mathrm{tr}(\rho)/2$. For $\alpha \in \Gamma$, we define the open subscheme $\mathrm{Rep}_2(\Gamma)_{u,\alpha}$ of $\mathrm{Rep}_2(\Gamma)_u$ by

$$\mathrm{Rep}_2(\Gamma)_{u,\alpha} := \{\rho \in \mathrm{Rep}_2(\Gamma)_u \mid \langle I_2, \rho(\alpha)\rangle \text{ generates a unipotent mold }\}.$$

Then $\mathrm{Rep}_2(\Gamma)_{u,\alpha}$ is a $\mathrm{PGL}_2[1/2]$-invariant open subscheme of $\mathrm{Rep}_2(\Gamma)_u$. The derivation $a_\alpha : \Gamma \to \Gamma(\mathrm{Rep}_2(\Gamma)_{u,\alpha}, \mathcal{O}_{\mathrm{Rep}_2(\Gamma)_{u,\alpha}})$ in Remark 5.7 induces the $A_1(\Gamma)[1/2]$-module homomorphism $\lambda^*(\Omega_\Gamma) \to \mathcal{O}_{\mathrm{Rep}_2(\Gamma)_{u,\alpha}}$, and hence we can define the morphism $\pi_\alpha : \mathrm{Rep}_2(\Gamma)_{u,\alpha} \to \mathrm{Ch}_2(\Gamma)_u$ over $\mathrm{Rep}_1(\Gamma)[1/2]$ associated to $\lambda^*(\Omega_\Gamma) \twoheadrightarrow \mathcal{O}_{\mathrm{Rep}_2(\Gamma)_{u,\alpha}}$. Gluing the morphisms $\{\pi_\alpha\}_{\alpha\in\Gamma}$, we have the morphism $\pi_{\Gamma,u} : \mathrm{Rep}_2(\Gamma)_u \to \mathrm{Ch}_2(\Gamma)_u$ over $\mathrm{Rep}_1(\Gamma)[1/2]$.

For $\alpha \in \Gamma$, we define the open subscheme $\mathrm{Ch}_2(\Gamma)_{u,\alpha}$ of $\mathrm{Ch}_2(\Gamma)_u$ by $\mathrm{Ch}_2(\Gamma)_{u,\alpha} := D(d\alpha) = \{d\alpha \neq 0\}$. From the definition of $\pi_{\Gamma,u}$, we see that $\pi_{\Gamma,u}^{-1}(\mathrm{Ch}_2(\Gamma)_{u,\alpha}) = \mathrm{Rep}_2(\Gamma)_{u,\alpha}$. For a $\mathbb{Z}[1/2]$-morphism $\psi : X \to \mathrm{Rep}_1(\Gamma)[1/2]$, there exists a one-to-one correspondence

$$\mathrm{Hom}_{\,\mathrm{Rep}_1(\Gamma)[1/2]}\,(X, \mathrm{Ch}_2(\Gamma)_{u,\alpha}) \cong \left\{\psi^*(\Omega_\Gamma) \twoheadrightarrow \mathcal{L} \to 0 \,\middle|\, \begin{array}{l} \mathcal{L} \text{ is a line bundle on } X \text{ and } \psi^*(d\alpha) \text{ is} \\ \text{nowhere vanishing as a section of } \mathcal{L} \end{array}\right\} \Big/ \sim .$$

Since $\mathcal{L}$ is generated by $\psi^*(d\alpha)$, $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$. Let $r : \Gamma \to \Gamma(X, \mathcal{O}_X)$ be the character associated to $\psi : X \to \mathrm{Rep}_1(\Gamma)[1/2]$. Regarding $\psi^*(d\alpha)$ as 1 of $\mathcal{O}_X$, we have the following:

$$\mathrm{Hom}_{\,\mathrm{Rep}_1(\Gamma)[1/2]}\,(X, \mathrm{Ch}_2(\Gamma)_{u,\alpha}) \cong \left\{ d \,\middle|\, \begin{array}{l} d : \Gamma \to \Gamma(X, \mathcal{O}_X) \text{ is a derivation with respect to } r \\ \text{such that } d(\alpha) = 1 \end{array}\right\}.$$

Remark that $\pi_{\Gamma,u} : \mathrm{Rep}_2(\Gamma)_u \to \mathrm{Ch}_2(\Gamma)_u$ and $\pi_\alpha : \mathrm{Rep}_2(\Gamma)_{u,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u,\alpha}$ are $\mathrm{PGL}_2[1/2]$-equivariant morphisms, where the actions of $\mathrm{PGL}_2[1/2]$ on $\mathrm{Ch}_2(\Gamma)_u$ and $\mathrm{Ch}_2(\Gamma)_{u,\alpha}$ are trivial.

Definition 5.15. For the free monoid $\Upsilon_1 = \langle \alpha_0 \rangle$ of rank 1, we say that the morphism $\pi_{\Upsilon_1,u} : \mathrm{Rep}_2(\Upsilon_1)_u \to \mathrm{Ch}_2(\Upsilon_1)_u$ is the *prototype* in the unipotent mold case. Remark that $\mathrm{Rep}_2(\Upsilon_1)_u = \mathrm{Rep}_2(\Upsilon_1)_{u,\alpha_0}$ and that $\mathrm{Ch}_2(\Upsilon_1)_u = \mathrm{Ch}_2(\Upsilon_1)_{u,\alpha_0}$.

By Theorem 4.17, $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ is a universal geometric quotient by PGL_2. Taking the base change of π by $\mathrm{Spec}\,\mathbb{Z}[1/2] \to \mathrm{Spec}\,\mathbb{Z}$, we have $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}[1/2] \to \mathrm{Ch}_2(\Upsilon_1)[1/2]$. Here we denote $X \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ by $X[1/2]$ for a $\mathbb{Z}$-scheme X. Let Z be the closed subscheme of $\mathrm{Ch}_2(\Upsilon_1)[1/2]$ defined by $m(\sigma_\Upsilon(\alpha_0)) = 0$. Since $\mathrm{Ch}_2(\Upsilon_1) = \mathrm{Spec}\,\mathbb{Z}[T, D]$, the affine ring of Z is isomorphic to $\mathbb{Z}[1/2, T]$. Note that $r(\cdot) = \mathrm{tr}(\sigma_{\Upsilon_1}(\cdot))/2$ gives a character of Υ_1 on Z and that $r(\alpha_0) = T/2$. Hence Z is isomorphic to $\mathrm{Ch}_2(\Upsilon_1)_u \cong \mathrm{Rep}_1(\Upsilon_1)[1/2]$. Taking the base change of $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}[1/2] \to \mathrm{Ch}_2(\Upsilon_1)[1/2]$ by $Z \hookrightarrow \mathrm{Ch}_2(\Upsilon_1)[1/2]$, we have $\pi_{\Upsilon_1,u} : \mathrm{Rep}_2(\Upsilon_1)_u \to \mathrm{Ch}_2(\Upsilon_1)_u$.

Here we introduce the following lemma without proof:

Lemma 5.16. *Let $X \to Y$ be a (universal) geometric quotient by G over S. For $S' \to S$, put $X_{S'} := X \times_S S'$, $Y_{S'} := Y \times_S S'$, and $G_{S'} := G \times_S S'$. Then $X_{S'} \to Y_{S'}$ is a (resp. universal) geometric quotient by $G_{S'}$ over S'.*

By the lemma above, we have:

Theorem 5.17. *The prototype $\pi_{\Upsilon_1,u} : \mathrm{Rep}_2(\Upsilon_1)_u \to \mathrm{Ch}_2(\Upsilon_1)_u$ is a universal geometric quotient by* $\mathrm{PGL}_2[1/2]$.

Let Γ be a group or a monoid. For $\alpha \in \Gamma$, we define the monoid homomorphism $\phi : \Upsilon_1 = \langle \alpha_0 \rangle \to \Gamma$ by $\alpha_0 \mapsto \alpha$. By restricting representations and derivations of Γ to those of Υ_1 through ϕ, we can obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}_2(\Gamma)_{u,\alpha} & \to & \mathrm{Ch}_2(\Gamma)_{u,\alpha} \\ \downarrow & & \downarrow \\ \mathrm{Rep}_2(\Upsilon_1)_u & \to & \mathrm{Ch}_2(\Upsilon_1)_u. \end{array}$$

Under this situation, we have the following lemma.

Lemma 5.18. *The above diagram gives a fibre product. In particular, the morphism $\mathrm{Rep}_2(\Gamma)_{u,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u,\alpha}$ is obtained by base change of the prototype.*

Proof. We claim that $\mathrm{Rep}_2(\Gamma)_{u,\alpha} \to \mathrm{Rep}_2(\Upsilon_1)_u \times_{\mathrm{Ch}_2(\Upsilon_1)_u} \mathrm{Ch}_2(\Gamma)_{u,\alpha}$ is an isomorphism. Let X be a $\mathbb{Z}[1/2]$-scheme. Assume that an X-valued point $\rho \in \mathrm{Rep}_2(\Gamma)_{u,\alpha}$ is sent to $(\rho', \sigma) \in \mathrm{Rep}_2(\Upsilon_1)_u \times_{\mathrm{Ch}_2(\Upsilon_1)_u} \mathrm{Ch}_2(\Gamma)_{u,\alpha}$.

We can regard the X-valued point $\sigma \in \mathrm{Ch}_2(\Gamma)_{u,\alpha}$ as a pair (r,d) such that $d : \Gamma \to \Gamma(X, \mathcal{O}_X)$ is a derivation with respect to a character $r : \Gamma \to \Gamma(X, \mathcal{O}_X)$ and $d(\alpha) = 1$. Since $\eta(\gamma) = d(\gamma)\eta(\alpha)$,

$$\rho(\gamma) = r(\gamma)I_2 + d(\gamma)\eta(\rho'(\alpha_0)) \tag{6}$$

for each $\gamma \in \Gamma$. Hence ρ is uniquely determined by (ρ', σ).

For an X-valued point $(\rho', \sigma) \in \mathrm{Rep}_2(\Upsilon_1)_u \times_{\mathrm{Ch}_2(\Upsilon_1)_u} \mathrm{Ch}_2(\Gamma)_{u,\alpha}$, we define the map $\rho : \Gamma \to \mathrm{M}_2(\Gamma(X, \mathcal{O}_X))$ by (6). From Lemma 5.8, we see that ρ is an X-valued point of $\mathrm{Rep}_2(\Gamma)_{u,\alpha}$. Then the X-valued point ρ is sent to $(\rho', \sigma) \in \mathrm{Rep}_2(\Upsilon_1)_u \times_{\mathrm{Ch}_2(\Upsilon_1)_u} \mathrm{Ch}_2(\Gamma)_{u,\alpha}$. By these discussion, the diagram gives a fibre product. □

Theorem 5.19. *The morphism $\pi_{\Gamma,u} : \mathrm{Rep}_2(\Gamma)_u \to \mathrm{Ch}_2(\Gamma)_u$ is a universal geometric quotient by $\mathrm{PGL}_2[1/2]$ for a group or a monoid Γ.*

Proof. For each $\alpha \in \Gamma$, $\pi_\alpha : \mathrm{Rep}_2(\Gamma)_{u,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u,\alpha}$ is a universal geometric quotient by $\mathrm{PGL}_2[1/2]$ because π_α is obtained by base change of the prototype. Hence this implies the statement. □

Remark 5.20. The morphism $\pi_{\Gamma,u} : \mathrm{Rep}_2(\Gamma)_u \to \mathrm{Ch}_2(\Gamma)_u$ is smooth and surjective. Indeed, the prototype $\pi_{\Upsilon_1,u} : \mathrm{Rep}_2(\Upsilon_1)_u \to \mathrm{Ch}_2(\Upsilon_1)_u$ is smooth and surjective because it is obtained by base change of $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ and π is smooth and surjective by Proposition 4.9.

Remark 5.21. For each point $x \in \mathrm{Ch}_2(\Gamma)_u$, there exists a local section $s_x : V_x \to \mathrm{Rep}_2(\Gamma)_u$ on a neighbourhood V_x of x such that $\pi_{\Gamma,u} \circ s_x = id_{V_x}$. Indeed, take $\alpha \in \Gamma$ such that $x \in \mathrm{Ch}_2(\Gamma)_{u,\alpha}$. The prototype $\mathrm{Rep}_2(\Upsilon_1)_u \to \mathrm{Ch}_2(\Upsilon_1)_u$ has a section s since it is obtained by base change of $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$, which has a section (it has been defined just before Proposition 4.12). By Lemma 5.18, $\pi_\alpha : \mathrm{Rep}_2(\Gamma)_{u,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u,\alpha}$ has a section $s_{\Gamma,\alpha}$. Hence we can take $\mathrm{Ch}_2(\Gamma)_{u,\alpha}$ as a neighbourhood V_x of x.

Lemma 5.22. *Let ρ_1, ρ_2 be representations with unipotent mold for a group (or a monoid) Γ on a scheme X over $\mathbb{Z}[1/2]$. Let $f_i : X \to \mathrm{Rep}_2(\Gamma)_u$ be the morphism associated to ρ_i for $i = 1, 2$. If $\pi_{\Gamma,u} \circ f_1 = \pi_{\Gamma,u} \circ f_2 : X \to \mathrm{Ch}_2(\Gamma)_u$, then for each $x \in X$ there exists $P_x \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood V_x of x such that $P_x^{-1}\rho_1 P_x = \rho_2$ on V_x.*

Proof. For $x \in X$, take $\alpha \in \Gamma$ such that $(\pi_{\Gamma,u} \circ f_1)(x) = (\pi_{\Gamma,u} \circ f_2)(x) \in \mathrm{Ch}_2(\Gamma)_{u,\alpha}$. We may assume that $f_i : X \to \mathrm{Rep}_2(\Gamma)_{u,\alpha}$ for $i = 1, 2$ from the beginning. By Remark 5.21, $\pi_\alpha : \mathrm{Rep}_2(\Gamma)_{u,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u,\alpha}$

has a section $s_{\Gamma,\alpha}$. Let ρ_3 be the representations with unipotent mold on X associated to $s_{\Gamma,\alpha} \circ \pi_{\Gamma,u} \circ f_1 = s_{\Gamma,\alpha} \circ \pi_{\Gamma,u} \circ f_2$. Then $\rho_i(\gamma) = r(\gamma)I_2 + d(\gamma)\eta(\rho_i(\alpha))$ for each $\gamma \in \Gamma$ and $i = 1, 2, 3$, where d is the derivation with respect to the character r associated to $\pi_{\Gamma,u} \circ f_1 = \pi_{\Gamma,u} \circ f_2$ such that $d(\alpha) = 1$. Note that $\rho_3(\alpha) = \begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix}$ and that $T = \mathrm{tr}(\rho_1(\alpha)) = \mathrm{tr}(\rho_2(\alpha)) = \mathrm{tr}(\rho_3(\alpha))$ and $D = T^2/4$. There exist $Q_1, Q_2 \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood V_x of x such that $Q_1^{-1}\rho_1(\alpha)Q_1 = \rho_3(\alpha)$ and $Q_2^{-1}\rho_2(\alpha)Q_2 = \rho_3(\alpha)$ by Lemma 4.8. Since $Q_1^{-1}\rho_1(\gamma)Q_1 = \rho_3(\gamma)$ and $Q_2^{-1}\rho_2(\gamma)Q_2 = \rho_3(\gamma)$ for each $\gamma \in \Gamma$, $(Q_1Q_2^{-1})^{-1}\rho_1(Q_1Q_2^{-1}) = \rho_2$ on V_x. This completes the proof. $\square$

Let us define $\mathcal{E}q\mathcal{U}_2(\Gamma)$ as the sheafification of the following contravariant functor with respect to Zariski topology:

$$\begin{array}{ccl} (\mathbf{Sch}/\mathbb{Z}[1/2])^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \{\rho \mid \text{ rep. with unipotent mold for } \Gamma \text{ on } X\}/\sim . \end{array}$$

By a *generalized representation with unipotent mold* for Γ on a scheme X, we understand pairs $\{(U_i, \rho_i)\}_{i\in I}$ of an open set U_i and a representation with unipotent mold $\rho_i : \Gamma \to \mathrm{M}_2(\Gamma(U_i, \mathcal{O}_X))$ satisfying the following two conditions:

(i) $\cup_{i\in I} U_i = X$,

(ii) for each $x \in U_i \cap U_j$, there exists $P_x \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood $V_x \subseteq U_i \cap U_j$ of x such that $P_x^{-1}\rho_i P_x = \rho_j$ on V_x.

Generalized representations with unipotent mold $\{(U_i, \rho_i)\}_{i\in I}$ and $\{(V_j, \sigma_j)\}_{j\in J}$ are called *equivalent* if $\{(U_i, \rho_i)\}_{i\in I} \cup \{(V_j, \sigma_j)\}_{j\in J}$ is a generalized representation with unipotent mold again. We easily see that $\mathcal{E}q\,\mathcal{U}_2(\Gamma)(X)$ is the set of equivalence classes of generalized representations with unipotent mold for Γ on a scheme X.

Theorem 5.23. *The scheme* $\mathrm{Ch}_2(\Gamma)_u$ *is a fine moduli scheme associated to the functor* $\mathcal{E}q\,\mathcal{U}_2(\Gamma)$ *for a group or a monoid* Γ*:*

$$\begin{array}{ccccl} \mathcal{E}q\,\mathcal{U}_2(\Gamma) & : & (\mathbf{Sch}/\mathbb{Z}[1/2])^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{r} \textit{gen. rep. with unipotent} \\ \textit{mold for } \Gamma \textit{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

In other words, $\mathrm{Ch}_2(\Gamma)_u$ *represents the functor* $\mathcal{E}q\,\mathcal{U}_2(\Gamma)$*. The moduli* $\mathrm{Ch}_2(\Gamma)_u$ *is separated over* $\mathbb{Z}[1/2]$*; if* Γ *is a finitely generated group or monoid, then* $\mathrm{Ch}_2(\Gamma)_u$ *is of finite type over* $\mathbb{Z}[1/2]$*.*

Proof. Since $\pi_{\Gamma,u} : \mathrm{Rep}_2(\Gamma)_u \to \mathrm{Ch}_2(\Gamma)_u$ is a $\mathrm{PGL}_2[1/2]$-equivariant morphism, we can define a canonical morphism $\mathcal{E}q\mathcal{U}_2(\Gamma) \to h_{\mathrm{Ch}_2(\Gamma)_u} :=$

$\mathrm{Hom}(-, \mathrm{Ch}_2(\Gamma)_u)$. Let us define a morphism $h_{\mathrm{Ch}_2(\Gamma)_u} \to \mathcal{E}q\,\mathcal{U}_2(\Gamma)$. Let $g \in h_{\mathrm{Ch}_2(\Gamma)_u}(X)$ with a $\mathbb{Z}[1/2]$-scheme X. For each $x \in X$, take $\alpha_x \in \Gamma$ such that $g(x) \in \mathrm{Ch}_2(\Gamma)_{u,\alpha_x}$. By using the section $s_{\Gamma,\alpha_x} : \mathrm{Ch}_2(\Gamma)_{u,\alpha_x} \to \mathrm{Rep}_2(\Gamma)_{u,\alpha_x}$ in Remark 5.21, we can define a representation with unipotent mold ρ_x on a neighbourhood U_x of x. By Lemma 5.22, we see that $\{(U_x, \rho_x)\}_{x\in X} \in \mathcal{E}q\,\mathcal{U}_2(\Gamma)(X)$ and that the morphism $h_{\mathrm{Ch}_2(\Gamma)_u} \to \mathcal{E}q\mathcal{U}_2(\Gamma)$ is well-defined. It is easy to see that $\mathrm{Ch}_2(\Gamma)_u$ represents the functor $\mathcal{E}q\,\mathcal{U}_2(\Gamma)$.

Since $\mathrm{Ch}_2(\Gamma)_u$ is defined as $\mathrm{Proj} S(\Omega_\Gamma)$, it is separated over $\mathbb{Z}[1/2]$. If Γ is finitely generated, then $\mathrm{Ch}_2(\Gamma)_u$ is of finite type over $\mathbb{Z}[1/2]$ by Remark 5.12. $\square$

Remark 5.24. Let A be an associative algebra over a commutative ring R over $\mathbb{Z}[1/2]$. For a 2-dimensional representation ρ of A on an R-scheme X, ρ is called a *representation with unipotent mold* if the subalgebra $\rho(A)$ of $\mathrm{M}_2(\mathcal{O}_X)$ generates a unipotent mold on X. In a similar way as group or monoid cases, we can define generalized representations with unipotent mold for A on an R-scheme X. The contravariant functor $\mathcal{E}q\,\mathcal{U}_2(A)$ from the category of R-schemes to the category of sets is defined as

$$\begin{array}{rcccl} \mathcal{E}q\,\mathcal{U}_2(A) & : & (\mathbf{Sch}/R)^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{r} \text{gen. rep. with unipotent} \\ \text{mold for } A \text{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

Then we can construct the fine moduli $\mathrm{Ch}_2(A)_u$ associated to $\mathcal{E}q\,\mathcal{U}_2(A)$ in the same way as Theorem 5.23 (for details, see Remark 5.25). The moduli $\mathrm{Ch}_2(A)_u$ is separated over R. If A is a finitely generated algebra over R, then $\mathrm{Ch}_2(A)_u$ is of finite type over R.

Remark 5.25. For an associative algebra A over a commutative ring R over $\mathbb{Z}[1/2]$, we can construct $\mathrm{Ch}_2(A)_u$ in the following way. We define the contravariant functor $\mathrm{Rep}_1(A)$ from the category of R-schemes to the category of sets by $X \mapsto \{\varphi : A \to \Gamma(X, \mathcal{O}_X) \mid R\text{-algebra hom. }\}$. The functor $\mathrm{Rep}_1(A)$ is representable by an affine scheme, and let us denote its coordinate ring by $A_1(A)$. Let $d : A \to A_1(A)$ be the universal R-algebra homomorphism. For an $A_1(A)$-module M, put

$$\mathrm{Der}(A, M) := \left\{ \delta : A \to M \,\middle|\, \begin{array}{c} \delta : \text{R-linear and for } a, b \in A, \\ \delta(ab) = d(a)\delta(b) + \delta(a)d(b) \end{array} \right\}.$$

The functor $\mathrm{Der}(A, -) : (A_1(A)\text{-}\mathbf{Mod}) \to (A_1(A)\text{-}\mathbf{Mod})$ defined by $M \mapsto \mathrm{Der}(A, M)$ is representable by some $A_1(A)$-module $\Omega_{A/R}$. Let $\mathrm{Rep}_2(A)$ be the representation variety of degree 2 for A over R, that is, the affine scheme representing the contravariant functor from the category of R-schemes to the category of sets which is defined by $X \mapsto \{2\text{-dim. rep. of } A \text{ on } X\}$. Let $\mathrm{Rep}_2(A)_u$ be the subscheme of $\mathrm{Rep}_2(A)$ consisting of representations with unipotent mold, and $\sigma : A \to \mathrm{M}_2(\Gamma(\mathrm{Rep}_2(A)_u, \mathcal{O}_{\mathrm{Rep}_2(A)_u}))$ the universal representation with unipotent mold. Then $\mathrm{tr}(\sigma(\cdot))/2 : A \to \Gamma(\mathrm{Rep}_2(A)_u, \mathcal{O}_{\mathrm{Rep}_2(A)_u})$ is an R-algebra homomorphism, and it defines a morphism $\mathrm{Rep}_2(A)_u \to \mathrm{Rep}_1(A)$. In a similar way as group or monoid cases, we can define a $\mathrm{Rep}_1(A)$-morphism $\pi : \mathrm{Rep}_2(A)_u \to \mathrm{Ch}_2(A)_u := \mathrm{Proj} S(\Omega_{A/R})$, where $S(\Omega_{A/R})$ is the symmetric algebra of $\Omega_{A/R}$ over $A_1(A)$. We can verify that π is a universal geometric quotient by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} R$ and that $\mathrm{Ch}_2(A)_u$ represents $\mathcal{E}q\,\mathcal{U}_2(A)$.

Remark 5.26. We have introduced the notion of generalized representations with unipotent mold for describing the moduli functors $\mathcal{E}q\mathcal{U}_2(\Gamma)$ and $\mathcal{E}q\mathcal{U}_2(A)$. However, the moduli functors can also be described as $\mathcal{E}q\mathcal{U}_2'(\Gamma)$ and $\mathcal{E}q\mathcal{U}_2'(A)$ by using the notion of representations generating sheaves of algebras which define unipotent molds. More precisely, see §8.

6. Unipotent mold over $\mathbb{F}_2$

In this section, all schemes are over $\mathrm{Spec}\,\mathbb{F}_2$ and all commutative rings are over $\mathbb{F}_2$. Recall that a rank 2 mold $\mathcal{A} \subseteq \mathrm{M}_2(\mathcal{O}_X)$ over an $\mathbb{F}_2$-scheme X is called *unipotent over* $\mathbb{F}_2$ if $\mathrm{tr}(s) = 0$ for each open subset $U \subseteq X$ and for each $s \in \mathcal{A}(U)$. We construct the moduli of representations with unipotent mold over $\mathbb{F}_2$.

Definition 6.1. Let X be an $\mathbb{F}_2$-scheme. Let $\mathcal{A} \subseteq \mathrm{M}_2(\mathcal{O}_X)$ be a unipotent mold over $\mathbb{F}_2$ on X. Let $U \subseteq X$ be an open subset of X. Suppose that $Z \in \mathcal{A}(U)$ satisfies $\mathcal{A}|_U = \mathcal{O}_U \cdot I_2 \oplus \mathcal{O}_U \cdot Z$. For each $Y \in \mathcal{A}(U)$, we set $Y = a_Z(Y)I_2 + b_Z(Y)Z$. We call $a_Z(Y), b_Z(Y) \in \mathcal{O}_X(U)$ the (a,b)-*coefficients* of Y with respect to Z.

Lemma 6.2. *Let U and Z be as in Definition 6.1. Assume that $\rho : \Gamma \to \mathrm{M}_2(\Gamma(X, \mathcal{O}_X))$ is a representation with unipotent mold $\mathcal{A}$ on X for a group or a monoid Γ. For each $\gamma \in \Gamma$, let us denote $a_Z(\rho(\gamma))$ and $b_Z(\rho(\gamma))$ by $a(\gamma)$ and $b(\gamma)$, respectively. Then for $\gamma, \delta \in \Gamma$, we have*

$$\begin{aligned} a(e) &= 1, \\ b(e) &= 0, \\ a(\gamma\delta) &= a(\gamma)a(\delta) + b(\gamma)b(\delta)\det Z, \\ b(\gamma\delta) &= a(\gamma)b(\delta) + b(\gamma)a(\delta). \end{aligned}$$

Proof. Since $\rho(e) = I_2 = 1 \cdot I_2 + 0 \cdot Z$, $a(e) = 1$ and $b(e) = 0$. By the Cayley-Hamilton theorem, $Z^2 - \mathrm{tr}(Z)Z + \det(Z)I_2 = 0$. Hence we have $Z^2 = -\det(Z)I_2 = \det(Z)I_2$ by $\mathrm{tr}(Z) = 0$. We see that

$$\begin{aligned}
\rho(\gamma\delta) &= \rho(\gamma)\rho(\delta) \\
&= (a(\gamma)I_2 + b(\gamma)Z)(a(\delta)I_2 + b(\delta)Z) \\
&= \{a(\gamma)a(\delta) + b(\gamma)b(\delta)\det(Z)\}I_2 + \{a(\gamma)b(\delta) + b(\gamma)a(\delta)\}Z.
\end{aligned}$$

Comparing the coefficients of $\rho(\gamma\delta) = a(\gamma\delta)I_2 + b(\gamma\delta)Z$, we obtain $a(\gamma\delta) = a(\gamma)a(\delta) + b(\gamma)b(\delta)\det Z$ and $b(\gamma\delta) = a(\gamma)b(\delta) + b(\gamma)a(\delta)$. □

Let R be an algebra over $\mathbb{F}_2$. Let $\rho : \Gamma \to \mathrm{M}_2(R)$ be a representation with unipotent mold $\mathcal{A}$ over $\mathbb{F}_2$ such that $\mathcal{A} = R \cdot I_2 \oplus R \cdot \rho(\alpha)$. For each $\gamma \in \Gamma$, we denote $a_{\rho(\alpha)}(\gamma), b_{\rho(\alpha)}(\gamma)$ by $a(\gamma), b(\gamma)$, respectively. Then $a(\cdot)$ and $b(\cdot)$ satisfy the formulas in Lemma 6.2, where $Z = \rho(\alpha)$. Furthermore, $a(\alpha) = 0$ and $b(\alpha) = 1$. Conversely, a character and (a, b)-coefficients give a representation with unipotent mold over $\mathbb{F}_2$:

Lemma 6.3. *Let $d : \Gamma \to R$ be a character. Let $\mathcal{A} = R \cdot I_2 \oplus R \cdot Z \subseteq \mathrm{M}_2(R)$ be a unipotent mold over $\mathbb{F}_2$ such that $\mathrm{tr}(Z) = 0$ and $\det(Z) = d(\alpha)$ for some $\alpha \in \Gamma$. Assume that $a : \Gamma \to R$, $b : \Gamma \to R$, and $\alpha \in \Gamma$ satisfy the equalities*

$$\begin{aligned}
a(\gamma\delta) &= a(\gamma)a(\delta) + b(\gamma)b(\delta)d(\alpha), \\
b(\gamma\delta) &= a(\gamma)b(\delta) + b(\gamma)a(\delta), \\
d(\gamma) &= a(\gamma)^2 + b(\gamma)^2 d(\alpha)
\end{aligned}$$

for each $\gamma, \delta \in \Gamma$. Furthermore, assume that $a(\alpha) = 0$ and that $b(\alpha) = 1$. Then the map

$$\begin{array}{rccl}
\rho : & \Gamma & \to & \mathrm{M}_2(R) \\
& \gamma & \mapsto & a(\gamma)I_2 + b(\gamma)Z
\end{array}$$

is a representation with unipotent mold $\mathcal{A}$ over $\mathbb{F}_2$ such that $\det(\rho(\gamma)) = d(\gamma)$ for each $\gamma \in \Gamma$.

Proof. First we show that $a(e) = 1$ and $b(e) = 0$. By the assumption,

$$\begin{aligned}
b(e) = b(e \cdot e) &= a(e)b(e) + b(e)a(e) \\
&= 2a(e)b(e) = 0
\end{aligned}$$

and

$$\begin{aligned}
a(e) = a(e \cdot e) &= a(e)a(e) + b(e)b(e)d(\alpha) \\
&= d(e) = 1.
\end{aligned}$$

Hence $\rho(e) = a(e)I_2 + b(e)Z = I_2$.

Next we show that $\rho(\gamma\delta) = \rho(\gamma)\rho(\delta)$ for each $\gamma, \delta \in \Gamma$. Since $Z^2 = \det(Z)I_2 = d(\alpha)I_2$,

$$\begin{aligned}\rho(\gamma)\rho(\delta) &= (a(\gamma)I_2 + b(\gamma)Z)(a(\delta)I_2 + b(\delta)Z) \\ &= \{a(\gamma)a(\delta) + b(\gamma)b(\delta)d(\alpha)\}I_2 + \{a(\gamma)b(\delta) + b(\gamma)a(\delta)\}Z \\ &= a(\gamma\delta)I_2 + b(\gamma\delta)Z \\ &= \rho(\gamma\delta).\end{aligned}$$

By Lemma 4.10, $\det \rho(\gamma) = \det(a(\gamma)I_2 + b(\gamma)Z) = a(\gamma)^2 \det I_2 + b(\gamma)^2 \det Z = a(\gamma)^2 + b(\gamma)^2 d(\alpha) = d(\gamma)$.

Finally, $\rho(\alpha) = 0 \cdot I_2 + 1 \cdot Z = Z$ implies that $\rho(\Gamma)$ generates $\mathcal{A}$. Hence ρ is a representation with unipotent mold $\mathcal{A}$ over $\mathbb{F}_2$. $\square$

Definition 6.4. Let $d : \Gamma \to R$ be a character in an $\mathbb{F}_2$-algebra R. For $a : \Gamma \to R$, $b : \Gamma \to R$ and $\alpha \in \Gamma$, we say that a and b are (a, b)-*coefficients* with respect to (d, α) if

$$\begin{aligned}&a(e) = 1, b(e) = 0, a(\alpha) = 0, b(\alpha) = 1, \\ a(\gamma\delta) &= a(\gamma)a(\delta) + b(\gamma)b(\delta)d(\alpha), \\ b(\gamma\delta) &= a(\gamma)b(\delta) + b(\gamma)a(\delta), \text{ and} \\ d(\gamma) &= a(\gamma)^2 + b(\gamma)^2 d(\alpha)\end{aligned}$$

hold for each $\gamma, \delta \in \Gamma$.

Definition 6.5. Set $\mathrm{Rep}_2(\Gamma)_{\mathbb{F}_2} := \mathrm{Rep}_2(\Gamma) \otimes_{\mathbb{Z}} \mathbb{F}_2$ and $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk2}/\mathbb{F}_2} := \mathrm{Rep}_2(\Gamma)_{\mathrm{rk2}} \otimes_{\mathbb{Z}} \mathbb{F}_2$. Let us define $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ as a closed subscheme of $\mathrm{Rep}_2(\Gamma)_{\mathrm{rk2}/\mathbb{F}_2}$ by

$$\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} := \{\rho \in \mathrm{Rep}_2(\Gamma)_{\mathrm{rk2}/\mathbb{F}_2} \mid \mathrm{tr}(\rho(\gamma)) = 0 \text{ for each } \gamma \in \Gamma \ \}.$$

For $\gamma \in \Gamma$, we define $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\gamma}$ as an open subscheme of $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ by

$$\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\gamma} := \left\{ \rho \in \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \ \middle| \ \begin{array}{l} I_2 \text{ and } \rho(\gamma) \text{ generate} \\ \text{a unipotent mold over } \mathbb{F}_2 \end{array} \right\}.$$

Note that

$$\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} = \bigcup_{\gamma \in \Gamma} \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\gamma}.$$

Definition 6.6. Set $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2} := \mathrm{Rep}_1(\Gamma) \otimes_{\mathbb{Z}} \mathbb{F}_2$. Let $A_1(\Gamma)_{\mathbb{F}_2}$ be the coordinate ring of $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$, and let $d : \Gamma \to A_1(\Gamma)_{\mathbb{F}_2}$ be the universal character of Γ. For $\alpha \in \Gamma$, we define the $A_1(\Gamma)_{\mathbb{F}_2}$-algebra $A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha}$ by $A_1(\Gamma)_{\mathbb{F}_2}[a(\gamma), b(\gamma) \mid \gamma \in \Gamma]/I$, where $a(\gamma), b(\gamma)$ are indeterminates for each $\gamma \in \Gamma$ and I is generated by

$$\begin{aligned} a(e)-1, b(e), a(\alpha), b(\alpha)-1 \\ a(\gamma\delta)-a(\gamma)a(\delta)-b(\gamma)b(\delta)d(\alpha) \\ b(\gamma\delta)-a(\gamma)b(\delta)-b(\gamma)a(\delta) \\ a(\gamma)^2-b(\gamma)^2 d(\alpha)-d(\gamma) \end{aligned}$$

for all $\gamma, \delta \in \Gamma$. We set $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} := \mathrm{Spec} A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha}$.

For $\alpha \in \Gamma$, $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ is a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-scheme. For a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-scheme X, denote by $\chi : \Gamma \to \Gamma(X, \mathcal{O}_X)$ the character of Γ associated to $X \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$. There exists a 1-1 correspondence

$$\mathrm{Hom}_{\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}}(X, \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}) \cong \left\{ (a,b) \;\middle|\; \begin{array}{l} a, b : \Gamma \to \Gamma(X, \mathcal{O}_X) \\ \text{are } (a,b)\text{-coefficients} \\ \text{with respect to } (\chi, \alpha) \end{array} \right\}.$$

Let $\sigma_{\Gamma,u/\mathbb{F}_2}$ and $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}$ be the universal representation with unipotent mold over $\mathbb{F}_2$ on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ and $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$, respectively. Put $d(\gamma) := \det(\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma))$ for $\gamma \in \Gamma$. By Lemma 6.2, $a_{\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)}(\cdot)$ and $b_{\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)}(\cdot)$ are (a,b)-coefficients with respect to (d, α) on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. Hence we have the morphism $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ associated to $a_{\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)}(\cdot)$ and $b_{\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)}(\cdot)$. Note that $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ is given by $\rho \mapsto \det(\rho)$. The group scheme $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ over $\mathbb{F}_2$ acts on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ by $\rho \mapsto P^{-1}\rho P$. The open subscheme $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ of $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ is $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-invariant for each $\alpha \in \Gamma$. Let $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ act on $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ trivially. Then we have:

Proposition 6.7. $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ *is* $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$*-equivariant.*

Proof. Let (ρ, P) be an X-valued point of $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \times (\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2)$ for an $\mathbb{F}_2$-scheme X. The representations ρ and $P^{-1}\rho P$ on X have the same determinants. For proving that ρ and $P^{-1}\rho P$ induce the same X-valued point of $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$, it only suffices to show that ρ and $P^{-1}\rho P$ have the same (a,b)-coefficients. We may assume that X is an affine scheme $\mathrm{Spec} R$ and that $P \in \mathrm{GL}_2(R)$. Note that $R[\rho(\Gamma)] = R \cdot I_2 + R \cdot \rho(\alpha) \subset \mathrm{M}_2(R)$ is a unipotent mold over $\mathbb{F}_2$. For each $\gamma \in \Gamma$, $\rho(\gamma) = a(\gamma)I_2 + b(\gamma)\rho(\alpha)$, where $a(\cdot)$ and $b(\cdot)$ are the (a,b)-coefficients of ρ. Multiplying the both sides by P^{-1} from the left and by P from the right, we have $P^{-1}\rho(\gamma)P = a(\gamma)I_2 + b(\gamma)P^{-1}\rho(\alpha)P$. Hence the (a,b)-coefficients of $P\rho P^{-1}$ with respect to $P^{-1}\rho(\alpha)P$ coincide with the (a,b)-coefficients of ρ with respect to $\rho(\alpha)$. This implies that (a,b)-coefficients are $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-invariant, which completes the proof. $\square$

Example 6.8. Let $\Upsilon_2 = \langle \alpha_1, \alpha_2 \rangle$ be the free monoid of rank 2. Let $\sigma := \sigma_{\Upsilon_2, u/\mathbb{F}_2, \alpha_1}$ denote the universal representation of Υ_2 on $\mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1}$. Then we can write $\sigma(\alpha_1) = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$, $\sigma(\alpha_2) = \begin{pmatrix} d & e \\ f & d \end{pmatrix}$, and

$$\mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1} = D(b) \cup D(c) \subset \mathrm{Spec}\, \mathbb{F}_2[a,b,c,d,e,f]/(bf - ce).$$

The (a,b)-coefficients of $\sigma(\alpha_2)$ with respect to $\sigma(\alpha_1)$ are given by $a(\alpha_2) = \dfrac{bd - ae}{b}$, $b(\alpha_2) = \dfrac{e}{b}$ on $D(b)$, and $a(\alpha_2) = \dfrac{cd - af}{c}$, $b(\alpha_2) = \dfrac{f}{c}$ on $D(c)$. For the universal matrix $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$,

$$P^{-1}\sigma(\alpha_1)P = \frac{1}{\Delta} \begin{pmatrix} a\Delta + brs - cpq & bs^2 - cq^2 \\ -br^2 + cp^2 & a\Delta - brs + cpq \end{pmatrix}$$

and

$$P^{-1}\sigma(\alpha_2)P = \frac{1}{\Delta} \begin{pmatrix} d\Delta + ers - fpq & es^2 - fq^2 \\ -er^2 + fp^2 & d\Delta - ers + fpq \end{pmatrix},$$

where $\Delta = ps - qr$. By direct calculation, we can verify that $a(\alpha_2)$ and $b(\alpha_2)$ are $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-invariant functions on $\mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1}$. Hence $\pi_{\Upsilon_2,u/\mathbb{F}_2,\alpha_1} : \mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1} \to \mathrm{Ch}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1}$ is $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-equivariant.

Remark 6.9. For any group Γ and for any $\alpha, \gamma \in \Gamma$, let us define the group homomorphism $\phi : \Upsilon_2 = \langle \alpha_1, \alpha_2 \rangle \to \Gamma$ by $\alpha_1 \mapsto \alpha$ and $\alpha_2 \mapsto \gamma$. Then ϕ induces a morphism $\phi^* : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1}$ by $\rho \mapsto \rho \circ \phi$. By Example 6.8, the (a,b)-coefficients $a(\alpha_2)$ and $b(\alpha_2)$ are $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-invariant functions on $\mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1}$. Let σ_Γ be the universal representation of Γ on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. Let $a_\alpha(\gamma)$ and $b_\alpha(\gamma)$ denote the (a,b)-coefficients of $\sigma_\Gamma(\gamma)$ with respect to $\sigma_\Gamma(\alpha)$. Note that $\phi_*(a(\alpha_2)) = a_\alpha(\gamma)$ and $\phi_*(b(\alpha_2)) = b_\alpha(\gamma)$, where ϕ_* denotes the ring homomorphism $\Gamma(\mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1}, \mathcal{O}_{\mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1}}) \to \Gamma(\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}, \mathcal{O}_{\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}})$ associated with $\phi^* : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_2(\Upsilon_2)_{u/\mathbb{F}_2,\alpha_1}$. Since ϕ^* is $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-equivariant, $a_\alpha(\gamma)$ and $b_\alpha(\gamma)$ are $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-invariant functions on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. Hence any (a,b)-coefficients are $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-invariant and $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ is $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-equivariant. This is another proof of Proposition 6.7.

Definition 6.10. For the free monoid $\Upsilon_1 = \langle \alpha_0 \rangle$ of rank 1, we say that the morphism $\pi_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0} : \mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0} \to \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ is the *prototype* in the unipotent mold over $\mathbb{F}_2$ case. Remark that $\mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2} = \mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ and that $\mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2} = \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$.

The coordinate ring $A_2(\Upsilon_1)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha_0}$ of $\mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ is isomorphic to $A_1(\Upsilon_1)_{\mathbb{F}_2}$. Indeed, $a(\alpha_0)=0$ and $b(\alpha_0)=1$. By induction on n, we can verify that $a(\alpha_0^n)=0, b(\alpha_0^n)=d(\alpha_0)^{(n-1)/2}$ for each positive odd integer n and that $a(\alpha_0^n)=d(\alpha_0)^{n/2}, b(\alpha_0^n)=0$ for each positive even integer n. Hence $A_2(\Upsilon_1)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha_0}\cong A_1(\Upsilon_1)_{\mathbb{F}_2}$ and $\mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}\cong \mathrm{Rep}_1(\Upsilon_1)_{\mathbb{F}_2}$.

By Theorem 4.17, $\pi:\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}\to\mathrm{Ch}_2(\Upsilon_1)$ is a universal geometric quotient by PGL_2. Taking the base change of π by $\mathrm{Spec}\,\mathbb{F}_2\to\mathrm{Spec}\,\mathbb{Z}$, we have $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2/\mathbb{F}_2}\to\mathrm{Ch}_2(\Upsilon_1)_{\mathbb{F}_2}$, where $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2/\mathbb{F}_2}:=\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2}\otimes_{\mathbb{Z}}\mathbb{F}_2$ and $\mathrm{Ch}_2(\Upsilon_1)_{\mathbb{F}_2}:=\mathrm{Ch}_2(\Upsilon_1)\otimes_{\mathbb{Z}}\mathbb{F}_2$. Let Z be the closed subscheme of $\mathrm{Ch}_2(\Upsilon_1)_{\mathbb{F}_2}$ defined by $\mathrm{tr}(\sigma_{\Upsilon_1}(\alpha_0))=0$. Since $\mathrm{Ch}_2(\Upsilon_1)=\mathrm{Spec}\,\mathbb{Z}[T,D]$, the affine ring of Z is isomorphic to $\mathbb{F}_2[D]$. Hence Z is isomorphic to $\mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}\cong\mathrm{Rep}_1(\Upsilon_1)_{\mathbb{F}_2}$. Taking the base change of $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2/\mathbb{F}_2}\to\mathrm{Ch}_2(\Upsilon_1)_{\mathbb{F}_2}$ by $Z\hookrightarrow\mathrm{Ch}_2(\Upsilon_1)_{\mathbb{F}_2}$, we have the prototype $\pi_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0}:\mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}\to\mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$.

By Lemma 5.16, we have:

Theorem 6.11. *The prototype* $\pi_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0}:\mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}\to\mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ *is a universal geometric quotient by* $\mathrm{PGL}_2\otimes_{\mathbb{Z}}\mathbb{F}_2$.

Let Γ be a group or a monoid. For $\alpha\in\Gamma$, we define the monoid homomorphism $\phi:\Upsilon_1=\langle\alpha_0\rangle\to\Gamma$ by $\alpha_0\mapsto\alpha$. By restricting representations and (a,b)-coefficients of Γ to those of Υ_1 through ϕ, we can obtain the following commutative diagram:

$$\begin{array}{ccc}\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} & \to & \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}\\ \downarrow & & \downarrow\\ \mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0} & \to & \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}.\end{array}$$

Under this situation, we have the following lemma.

Lemma 6.12. *The above diagram gives a fibre product. In particular, the morphism* $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}\to\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ *is obtained by base change of the prototype.*

Proof. Set $Z:=\mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}\times_{\mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}}\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. We claim that $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}\to Z$ is an isomorphism. Let X be an $\mathbb{F}_2$-scheme. Assume that an X-valued point $\rho\in\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ is sent to $(\rho',\sigma)\in Z$. We can regard the X-valued point $\sigma\in\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ as a pair $(d,(a,b))$ such that $a,b:\Gamma\to\Gamma(X,\mathcal{O}_X)$ are (a,b)-coefficients with respect to (d,α), where $d(\cdot):=\det(\rho(\cdot)):\Gamma\to\Gamma(X,\mathcal{O}_X)$. Since

$$\rho(\gamma)=a(\gamma)I_2+b(\gamma)\rho'(\alpha_0)\tag{7}$$

for each $\gamma\in\Gamma$, ρ is uniquely determined by (ρ',σ).

For an X-valued point $(\rho',\sigma)\in Z$, we define the map $\rho:\Gamma\to\mathrm{M}_2(\Gamma(X,\mathcal{O}_X))$ by (7). From Lemma 6.3, we see that ρ is an X-valued point of $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. Then the X-valued point ρ is sent to $(\rho', \sigma) \in Z$. By these discussion, we see that $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to Z$ is an isomorphism, and hence that the diagram gives a fibre product. □

Theorem 6.13. *The morphism* $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ *is a universal geometric quotient by* $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ *for each* $\alpha \in \Gamma$.

Proof. The statement follows from that $\pi_{\Gamma,u/\mathbb{F}_2,\alpha}$ is obtained by base change of the prototype. □

Let $\alpha, \beta \in \Gamma$. Let $U_{\alpha,\beta} \subseteq \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ be the open subscheme defined by $\{b(\beta) \neq 0\}$. The inverse image $\pi^{-1}_{\Gamma,u/\mathbb{F}_2,\alpha}(U_{\alpha,\beta})$ by $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ coincides with $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \cap \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\beta}$. Then $\pi^{-1}_{\Gamma,u/\mathbb{F}_2,\alpha}(U_{\alpha,\beta}) = \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \cap \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\beta} \to U_{\alpha,\beta}$ is a universal geometric quotient by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$. Hence $U_{\alpha,\beta} \cong U_{\beta,\alpha}$, and let us denote the canonical isomorphism by $\varphi_{\alpha,\beta} : U_{\alpha,\beta} \to U_{\beta,\alpha}$. Note that $\varphi_{\beta,\alpha} = \varphi^{-1}_{\alpha,\beta}$. For $\alpha, \beta, \gamma \in \Gamma$, $\varphi_{\alpha,\beta}(U_{\alpha,\beta} \cap U_{\alpha,\gamma}) = U_{\beta,\alpha} \cap U_{\beta,\gamma}$ and $\varphi_{\alpha,\gamma} = \varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta}$ on $U_{\alpha,\beta} \cap U_{\alpha,\gamma}$. Gluing the schemes $\{\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}\}_{\alpha\in\Gamma}$, we obtain a scheme, which we call $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ (for example, see [6, Chap. II, Ex. 2.12]). Gluing $\{\pi_{\Gamma,u/\mathbb{F}_2,\alpha}\}_{\alpha\in\Gamma}$, we also obtain $\pi_{\Gamma,u/\mathbb{F}_2} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$.

Corollary 6.14. *The morphism* $\pi_{\Gamma,u/\mathbb{F}_2} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *is a universal geometric quotient by* $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$.

Remark 6.15. By Definition 6.6, $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ is a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-scheme for each $\alpha \in \Gamma$. Let $d_\alpha : \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ be the canonical morphism for each $\alpha \in \Gamma$. We can obtain a morphism $d : \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ by gluing the canonical morphisms $\{d_\alpha\}_{\alpha\in\Gamma}$. Hence $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is also a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-scheme. Let us denote by $\det : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ the morphism corresponding to the character $\det(\sigma_{\Gamma,u/\mathbb{F}_2}(\cdot)))$. Then $d \circ \pi_{\Gamma,u/\mathbb{F}_2} = \det$.

The open subscheme $U_{\alpha,\beta} \subseteq \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ is affine and its coordinate ring is isomorphic to the localization $A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha}[b_\alpha(\beta)^{-1}]$ of the coordinate ring $A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha}$ of $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ by $b_\alpha(\beta)^{-1}$. Here we denote by a_α, b_α the (a, b)-coefficients of the universal representations $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}$ with respect to $(\det(\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}), \alpha)$. Let $\varphi^*_{\alpha,\beta} : A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\beta}[b_\beta(\alpha)^{-1}] \to A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha}[b_\alpha(\beta)^{-1}]$ denote by the ring isomorphism associated to $\varphi_{\alpha,\beta} : U_{\alpha,\beta} \to U_{\beta,\alpha}$. Then $\varphi^*_{\alpha,\beta}(a_\beta(\gamma)) = a_\alpha(\gamma) + a_\beta(\alpha) b_\alpha(\gamma)$ and $\varphi^*_{\alpha,\beta}(b_\beta(\gamma)) = b_\alpha(\gamma) b_\beta(\alpha)$ for each $\gamma \in \Gamma$. Note that $b_\beta(\alpha) = b_\alpha(\beta)^{-1}$ and $a_\beta(\alpha) = -a_\alpha(\beta) b_\alpha(\beta)^{-1}$ on $U_{\alpha,\beta} \cong U_{\beta,\alpha}$. Since $A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha} \otimes_{\mathbb{F}_2} A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\beta} \to$

$A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha}[b_\alpha(\beta)^{-1}]$ is a surjective ring homomorphism, the diagonal morphism $U_{\alpha,\beta} \xrightarrow{\Delta} \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \times_{\mathbb{F}_2} \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\beta}$ is a closed immersion. Hence we have:

Proposition 6.16. *For a group or a monoid* Γ, $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *is separated over* $\mathbb{F}_2$.

Remark 6.17. The morphism $\pi_{\Gamma,u/\mathbb{F}_2} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is smooth and surjective for each group or monoid Γ. Indeed, the prototype $\pi_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0} : \mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0} \to \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ is smooth and surjective because it is obtained by base change of $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \to \mathrm{Ch}_2(\Upsilon_1)$ and π is smooth and surjective by Proposition 4.9. Hence $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} = \pi^{-1}_{\Gamma,u/\mathbb{F}_2}(\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}) \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ is smooth and surjective for each $\alpha \in \Gamma$. Therefore, so is $\pi_{\Gamma,u/\mathbb{F}_2}$.

Remark 6.18. For each point $x \in \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$, there exists a local section $s_x : V_x \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ on a neighbourhood V_x of x such that $\pi_{\Gamma,u/\mathbb{F}_2} \circ s_x = id_{V_x}$. Indeed, take $\alpha \in \Gamma$ such that $x \in \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. The prototype $\mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0} \to \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ has a section s since it is obtained by base change of $\mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk2}} \to \mathrm{Ch}_2(\Upsilon_1)$, which has a section (it has been defined just before Proposition 4.12). By Lemma 6.12, we see that $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ has a section $s_{\Gamma,\alpha}$. Hence we can take $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ as a neighbourhood V_x of x.

Lemma 6.19. *Let* ρ_1, ρ_2 *be representations with unipotent mold over* $\mathbb{F}_2$ *for a group (or a monoid)* Γ *on a scheme* X *over* $\mathbb{F}_2$. *Let* $f_i : X \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ *be the morphism associated to* ρ_i *for* $i = 1, 2$. *If* $\pi_{\Gamma,u/\mathbb{F}_2} \circ f_1 = \pi_{\Gamma,u/\mathbb{F}_2} \circ f_2 : X \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$, *then for each* $x \in X$ *there exists* $P_x \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ *on a neighbourhood* V_x *of* x *such that* $P_x^{-1}\rho_1 P_x = \rho_2$ *on* V_x.

Proof. For $x \in X$, take $\alpha \in \Gamma$ such that $(\pi_{\Gamma,u/\mathbb{F}_2} \circ f_1)(x) = (\pi_{\Gamma,u/\mathbb{F}_2} \circ f_2)(x) \in \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. We may assume that $f_i : X \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ for $i = 1, 2$ from the beginning. By Remark 6.18, $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ has a section $s_{\Gamma,\alpha}$. Let ρ_3 be the representations with unipotent mold over $\mathbb{F}_2$ on X associated to $s_{\Gamma,\alpha} \circ \pi_{\Gamma,u/\mathbb{F}_2} \circ f_1 = s_{\Gamma,\alpha} \circ \pi_{\Gamma,u/\mathbb{F}_2} \circ f_2$. Then $\rho_i(\gamma) = a(\gamma)I_2 + b(\gamma)\rho_i(\alpha)$ for each $\gamma \in \Gamma$ and $i = 1, 2, 3$, where (a, b) is the (a, b)-coefficients with respect to (d, α) and $d(\cdot) := \det(\rho_1(\cdot)) = \det(\rho_2(\cdot)) = \det(\rho_3(\cdot))$.

Note that $\rho_3(\alpha) = \begin{pmatrix} 0 & -D \\ 1 & 0 \end{pmatrix}$ and that $D = d(\alpha)$. There exist $Q_1, Q_2 \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood V_x of x such that $Q_1^{-1}\rho_1(\alpha)Q_1 = \rho_3(\alpha)$ and $Q_2^{-1}\rho_2(\alpha)Q_2 = \rho_3(\alpha)$ by Lemma 4.8. Then

$Q_1^{-1}\rho_1(\gamma)Q_1 = \rho_3(\gamma)$ and $Q_2^{-1}\rho_2(\gamma)Q_2 = \rho_3(\gamma)$ for each $\gamma \in \Gamma$, and hence $(Q_1Q_2^{-1})^{-1}\rho_1(Q_1Q_2^{-1}) = \rho_2$ on V_x. This completes the proof. $\square$

Let us define $\mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$ as the sheafification of the following contravariant functor with respect to Zariski topology:

$$\begin{array}{ccc} (\mathbf{Sch}/\mathbb{F}_2)^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \left\{ \rho \left| \begin{array}{l} \text{rep. with unip. mold} \\ \text{over } \mathbb{F}_2 \text{ for } \Gamma \text{ on } X \end{array} \right. \right\} \Big/ \sim . \end{array}$$

By a *generalized representation with unipotent mold over* $\mathbb{F}_2$ for Γ on an $\mathbb{F}_2$-scheme X, we understand pairs $\{(U_i, \rho_i)\}_{i\in I}$ of an open set U_i and a representation $\rho_i : \Gamma \to \mathrm{M}_2(\Gamma(U_i, \mathcal{O}_X))$ with unipotent mold over $\mathbb{F}_2$ satisfying the following two conditions:

(i) $\cup_{i\in I} U_i = X$,

(ii) for each $x \in U_i \cap U_j$, there exists $P_x \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood $V_x \subseteq U_i \cap U_j$ of x such that $P_x^{-1}\rho_i P_x = \rho_j$ on V_x.

Generalized representations $\{(U_i, \rho_i)\}_{i\in I}$ and $\{(V_j, \sigma_j)\}_{j\in J}$ with unipotent mold over $\mathbb{F}_2$ are called *equivalent* if $\{(U_i, \rho_i)\}_{i\in I} \cup \{(V_j, \sigma_j)\}_{j\in J}$ is a generalized representation with unipotent mold over $\mathbb{F}_2$ again. We easily see that $\mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}(X)$ is the set of equivalence classes of generalized representations with unipotent mold over $\mathbb{F}_2$ for Γ on an $\mathbb{F}_2$-scheme X.

Theorem 6.20. *The scheme* $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *is a fine moduli scheme associated to the functor* $\mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$ *for a group or a monoid* Γ*:*

$$\begin{array}{ccccc} \mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2} & : & (\mathbf{Sch}/\mathbb{F}_2)^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{c} \textit{gen. rep. with unipotent} \\ \textit{mold over } \mathbb{F}_2 \textit{ for } \Gamma \textit{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

In other words, $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *represents the functor* $\mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$*. The moduli* $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *is separated over* $\mathbb{F}_2$*; if* Γ *is a finitely generated group or monoid, then* $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *is of finite type over* $\mathbb{F}_2$.

Proof. Since $\pi_{\Gamma,u/\mathbb{F}_2} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is a $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-equivariant morphism, we can define a canonical morphism $\mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2} \to h_{\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}} := \mathrm{Hom}(-, \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2})$. We define a morphism $h_{\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}} \to \mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$ as follows. Let $g \in h_{\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}}(X)$ with an $\mathbb{F}_2$-scheme X. For each $x \in X$, take $\alpha_x \in \Gamma$ such that $g(x) \in \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha_x}$. By using the section $s_{\Gamma,\alpha_x} : \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha_x} \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha_x}$ in Remark 6.18, we can define a representation ρ_x with unipotent mold over $\mathbb{F}_2$ on a neighbourhood U_x of x. By Lemma 6.19, we see that $\{(U_x, \rho_x)\}_{x\in X} \in \mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}(X)$ and that the morphism $h_{\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}} \to \mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$ is well-defined. It is easy to see that $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ represents the functor $\mathcal{E}q\,\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$.

By Proposition 6.16, $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is separated over $\mathbb{F}_2$. If Γ is finitely generated, then we can verify that $A_2(\Gamma)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha}$ is a finitely generated algebra over $\mathbb{F}_2$ in a similar way as Remark 5.12. Let $S = \{\alpha_1, \dots, \alpha_n\}$ be a set of generators of Γ. Then $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is covered by finitely many affine open subschemes $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha_i}(1 \leq i \leq n)$. Hence $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is of finite type over $\mathbb{F}_2$. $\square$

In the following Example 6.21, we describe $\mathrm{Ch}_2(\Upsilon_m)_{u/\mathbb{F}_2}$ for the free monoid $\Upsilon_m = \langle \alpha_1, \dots, \alpha_m \rangle$ of rank m. This description has been inspired by the referee.

Example 6.21. Let us describe $\mathrm{Ch}_2(\Upsilon_m)_{u/\mathbb{F}_2}$ for the free monoid $\Upsilon_m = \langle \alpha_1, \dots, \alpha_m \rangle$ of rank m. Put $C(m) := \mathrm{Ch}_2(\Upsilon_m)_{u/\mathbb{F}_2}$ and $C(m)_i := \mathrm{Ch}_2(\Upsilon_m)_{u/\mathbb{F}_2,\alpha_i}$ for $1 \leq i \leq m$. Let us denote by $A(m)_i$ the $A_1(\Upsilon_m)_{\mathbb{F}_2}$-algebra $A_2(\Upsilon_m)^{\mathrm{Ch}}_{u/\mathbb{F}_2,\alpha_i}$ for $1 \leq i \leq m$ in Definition 6.6. We can write $A_1(\Upsilon_m)_{\mathbb{F}_2} = \mathbb{F}_2[d(\alpha_1), \dots, d(\alpha_m)]$ and

$$A(m)_i = \mathbb{F}_2[d(\alpha_j), a_i(\alpha_j), b_i(\alpha_j) \mid 1 \leq j \leq m]/I(m)_i,$$

where $I(m)_i$ is the ideal of $\mathbb{F}_2[d(\alpha_j), a_i(\alpha_j), b_i(\alpha_j) \mid 1 \leq j \leq m]$ generated by $a_i(\alpha_i)$, $b_i(\alpha_i) - 1$, and $d(\alpha_j) - a_i(\alpha_j)^2 - b_i(\alpha_j)^2 d(\alpha_i)$ for $1 \leq j \leq m$. Note that

$$\begin{aligned} C(m)_i \cong \mathbb{A}^{2m-1}_{\mathbb{F}_2} = \{(&a_i(\alpha_1), \dots, a_i(\alpha_{i-1}), a_i(\alpha_{i+1}), \dots, a_i(\alpha_m), \\ &b_i(\alpha_1), \dots, b_i(\alpha_{i-1}), b_i(\alpha_{i+1}), \dots, b_i(\alpha_m), d(\alpha_i)) \in \mathbb{A}^{2m-1}_{\mathbb{F}_2}\}. \end{aligned}$$

We set $U_{ij} := \{b_i(\alpha_j) \neq 0\} \subset C(m)_i = \mathrm{Spec} A(m)_i$. For $1 \leq i \neq j \leq m$, the isomorphism $\varphi_{ij} : U_{ij} \to U_{ji}$ is given by the $\mathbb{F}_2$-algebra isomorphism $\varphi^*_{ij} : A(m)_j[b_j(\alpha_i)^{-1}] \to A(m)_i[b_i(\alpha_j)^{-1}]$ which is defined by $\varphi^*_{ij}(a_j(\alpha_k)) = a_i(\alpha_k) + b_i(\alpha_k) a_i(\alpha_j)/b_i(\alpha_j)$, $\varphi^*_{ij}(b_j(\alpha_k)) = b_i(\alpha_k)/b_i(\alpha_j)$, and $\varphi^*_{ij}(d(\alpha_k)) = d(\alpha_k)$ for $1 \leq k \leq m$.

On the other hand, let us define the closed subvariety $D(m)$ of $\mathbb{P}^{m^2+m-1}_{\mathbb{F}_2} \times \mathbb{A}^m_{\mathbb{F}_2}$ over $\mathbb{F}_2$ in the following way:

$$\begin{aligned} D(m) := \{([a_{ij} : b_1 : \cdots : b_m]_{1 \leq i,j \leq m}, (d_1, \dots, d_m)) \in \mathbb{P}^{m^2+m-1}_{\mathbb{F}_2} \times \mathbb{A}^m_{\mathbb{F}_2} \mid \\ a_{ji} = a_{ij},\ a_{ii} = 0\ (1 \leq i, j \leq m), \\ a_{ij}^2 + b_i^2 d_j + b_j^2 d_i = 0\ (1 \leq i, j \leq m), \\ a_{ij} b_k + a_{jk} b_i + a_{ki} b_j = 0\ (1 \leq i, j, k \leq m)\}. \end{aligned}$$

(Note that $D(m)$ can also be defined as a closed subvariety of $\mathbb{P}^{\frac{m(m+1)}{2}-1}_{\mathbb{F}_2} \times \mathbb{A}^m_{\mathbb{F}_2}$ by using homogeneous coordinates $\{a_{ij}\}_{1 \leq i < j \leq m}$ and $\{b_i\}_{1 \leq i \leq m}$.)

Put $D(m)_i := \{b_i \neq 0\} \subset D(m)$ for $1 \leq i \leq m$. By using inhomogeneous coordinates $\overline{a}_{jk} = a_{jk}/b_i$ and $\overline{b}_j = b_j/b_i$ for $1 \leq j, k \leq m$, we easily see that

$$\begin{aligned} & D(m)_i \cong \mathbb{A}^{2m-1}_{\mathbb{F}_2} \\ = & \{(\overline{a}_{i1}, \ldots, \overline{a}_{i,i-1}, \overline{a}_{i,i+1}, \ldots, \overline{a}_{im}, \overline{b}_1, \ldots, \overline{b}_{i-1}, \overline{b}_{i+1}, \ldots, \overline{b}_m, d_i) \in \mathbb{A}^{2m-1}_{\mathbb{F}_2}\}. \end{aligned}$$

Note that $D(m) = \cup_{i=1}^m D(m)_i$.

Let us define the $\mathbb{F}_2$-isomorphism $\psi_i : C(m)_i \to D(m)_i \subset D(m)$ by the $\mathbb{F}_2$-algebra isomorphism $\psi_i^* : A(D(m)_i) \to A(m)_i$ which is defined by $\psi_i^*(\overline{a}_{ij}) = a_i(\alpha_j)$, $\psi_i^*(\overline{b}_j) = b_i(\alpha_j)$ for $j \neq i$, and $\psi_i^*(d_i) = d(\alpha_i)$, where $A(D(m)_i)$ is the coordinate ring of $D(m)_i$. We can glue $\{\psi_i : C(m)_i \to D(m)_i\}_{1 \leq i \leq m}$ (remark that -1 equals to 1 in characteristic 2 for the verification), and hence we have an isomorphism $\psi : C(m) \to D(m)$. Note that $\mathrm{Ch}_2(\Upsilon_m)_{u/\mathbb{F}_2} = C(m) \cong D(m)$ is a $(2m-1)$-dimensional smooth irreducible variety over $\mathbb{F}_2$.

Let ρ_i be the representation of Υ_m on $D(m)_i \cong C(m)_i$ defined by

$$\rho_i(\alpha_j) = \begin{pmatrix} \overline{a}_{ij} & \overline{b}_j d_i \\ \overline{b}_j & \overline{a}_{ij} \end{pmatrix}$$

for $1 \leq j \leq m$. Then $\{(C(m)_i, \rho_i)\}_{1 \leq i \leq m}$ is the universal equivalence class of generalized representations with unipotent mold over $\mathbb{F}_2$ of Υ_m on $\mathrm{Ch}_2(\Upsilon_m)_{u/\mathbb{F}_2}$.

Remark 6.22. Let A be an associative algebra over a commutative ring R over $\mathbb{F}_2$. For a 2-dimensional representation ρ of A on an R-scheme X, ρ is called a *representation with unipotent mold over* $\mathbb{F}_2$ if the subalgebra $\rho(A)$ of $\mathrm{M}_2(\mathcal{O}_X)$ generates a unipotent mold over $\mathbb{F}_2$ on X. In a similar way as group or monoid cases, we can define generalized representations with unipotent mold over $\mathbb{F}_2$ for A on an R-scheme X. The contravariant functor $\mathcal{E}q\,\mathcal{U}_2(A)_{\mathbb{F}_2}$ from the category of R-schemes to the category of sets is defined as

$$\begin{array}{rccl} \mathcal{E}q\,\mathcal{U}_2(A)_{\mathbb{F}_2} & : & (\mathbf{Sch}/R)^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{c} \text{gen. rep. with unipotent} \\ \text{mold over } \mathbb{F}_2 \text{ for } A \text{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

We can construct the fine moduli $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ associated to $\mathcal{E}q\,\mathcal{U}_2(A)_{\mathbb{F}_2}$ in the same way as Theorem 6.20 (for details, see Remark 6.23). The moduli $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ is separated over R. If A is a finitely generated algebra over R, then $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ is of finite type over R.

Remark 6.23. For an associative algebra A over a commutative ring R over $\mathbb{F}_2$, we can construct $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ in the following way. We define the contravariant functor $\mathrm{Rep}'_1(A)$ from the category of R-schemes to the category of sets by

$$\mathrm{Rep}'_1(A)(X) := \left\{ d : A \to \Gamma(X, \mathcal{O}_X) \,\middle|\, \begin{array}{c} d \text{ is a ring homomorphism and} \\ d(\alpha a) = \alpha^2 d(a) \text{ for } a \in A, \alpha \in R \end{array} \right\}$$

for each R-scheme X. The functor $\mathrm{Rep}'_1(A)$ is representable by an affine scheme, and let us denote its coordinate ring by $A'_1(A)$. Let $d : A \to A'_1(A)$ be the universal ring homomorphism. Let $\mathrm{Rep}_2(A)$ be the representation variety of degree 2 for A over R introduced in Remark 5.25. Let $\mathrm{Rep}_2(A)_{u/\mathbb{F}_2}$ be the subscheme of $\mathrm{Rep}_2(A)$ consisting of representations with unipotent mold over $\mathbb{F}_2$, and $\sigma_{A,u/\mathbb{F}_2} : A \to \mathrm{M}_2(\Gamma(\mathrm{Rep}_2(A)_{u/\mathbb{F}_2}, \mathcal{O}_{\mathrm{Rep}_2(A)_{u/\mathbb{F}_2}}))$ the universal representation with unipotent mold over $\mathbb{F}_2$. Note that $\det \sigma_{A,u/\mathbb{F}_2} : A \to \Gamma(\mathrm{Rep}_2(A)_{u/\mathbb{F}_2}, \mathcal{O}_{\mathrm{Rep}_2(A)_{u/\mathbb{F}_2}})$ gives a morphism $\mathrm{Rep}_2(A)_{u/\mathbb{F}_2} \to \mathrm{Rep}'_1(A)$. For $c \in A$, define the open subscheme $\mathrm{Rep}_2(A)_{u/\mathbb{F}_2,c} := \{\langle I_2, \sigma_{A,u/\mathbb{F}_2}(c)\rangle$ is a unipotent mold over $\mathbb{F}_2$ $\}$ of $\mathrm{Rep}_2(A)_{u/\mathbb{F}_2}$. For a scheme X over $A'_1(A)$ and for $c \in A$, we say that $a, b : A \to \Gamma(X, \mathcal{O}_X)$ are (a,b)-*coefficients* with respect to (d, c) on X if a, b are R-linear maps satisfying

$$\begin{aligned} a(1) = 1, a(c) = 0, b(1) = 0, b(c) = 1,& \\ a(c_1c_2) = a(c_1)a(c_2) + b(c_1)b(c_2)d(c),& \\ b(c_1c_2) = a(c_1)b(c_2) + b(c_1)a(c_2),& \\ a(c_1)^2 + b(c_1)^2 d(c) = d(c_1),& \end{aligned}$$

for all $c_1, c_2 \in A$. Here $d : A \to \Gamma(X, \mathcal{O}_X)$ denotes the ring homomorphism associated to $X \to \mathrm{Rep}'_1(A)$. There exists a commutative ring $A_2(A)^{\mathrm{Ch}}_{u/\mathbb{F}_2,c}$ over $A'_1(A)$ such that $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2,c} := \mathrm{Spec} A_2(A)^{\mathrm{Ch}}_{u/\mathbb{F}_2,c}$ represents the functor corresponding X to the set of (a,b)-coefficients with respect to (d, c) on X for each scheme X over $A'_1(A)$. In a similar way as group or monoid cases, we can define a $\mathrm{Rep}'_1(A)$-morphism $\pi_c : \mathrm{Rep}_2(A)_{u/\mathbb{F}_2,c} \to \mathrm{Ch}_2(A)_{u/\mathbb{F}_2,c}$. We see that π_c is a universal geometric quotient by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} R$. Gluing schemes $\{\mathrm{Ch}_2(A)_{u/\mathbb{F}_2,c}\}_{c\in A}$, we have a scheme $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ over $\mathrm{Rep}'_1(A)$. Gluing $\{\pi_c\}_{c\in A}$, we also have a morphism $\pi : \mathrm{Rep}_2(A)_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ over $\mathrm{Rep}'_1(A)$. Hence π is a universal geometric quotient by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} R$ and $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ represents $\mathcal{E}q\,\mathcal{U}_2(A)_{\mathbb{F}_2}$.

Remark 6.24. We have introduced the notion of generalized representations with unipotent mold over $\mathbb{F}_2$ for describing the moduli functors $\mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$ and $\mathcal{E}q\mathcal{U}_2(A)_{\mathbb{F}_2}$. However, the moduli functors can also be described as $\mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2}$ and $\mathcal{E}q\mathcal{U}'_2(A)_{\mathbb{F}_2}$ by using the notion of representations generating sheaves of algebras which define unipotent molds over $\mathbb{F}_2$. More precisely, see §8.

7. Another approach for unipotent molds over $\mathbb{F}_2$

In this section, we construct the moduli scheme $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ in a different way from §6. When we take the quotient of $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$, we need to introduce the notion of (a,b)-coefficients because there exist no eigenvalue of $\sigma_{\Gamma,u/\mathbb{F}_2}(\gamma)$ on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ in general, where $\sigma_{\Gamma,u/\mathbb{F}_2}$ is the universal representation on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$. However, we can obtain eigenvalues of $\sigma_{\Gamma,u/\mathbb{F}_2}(\gamma)$ by taking the pull-back of $\sigma_{\Gamma,u/\mathbb{F}_2}$ by a faithfully flat finite morphism $p : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$. Then by discussing derivations we can construct a universal geometric quotient $\tilde{\pi}_{\Gamma,u/\mathbb{F}_2} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ in the same way as the unipotent mold ($ch \neq 2$) case in §5. Considering the "descent" of $\tilde{\pi}_{\Gamma,u/\mathbb{F}_2} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$, we have a universal geometric quotient $\pi_{\Gamma,u/\mathbb{F}_2} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$. In this section, we will use the same notation as §6. Without Lemma 6.12, we will prove Theorem 6.13. It should be pointed out that this section was inspired by the referee.

Let Γ be a group or a monoid. For $\alpha \in \Gamma$, let us consider the scheme $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ over $\mathbb{F}_2$. Recall that

$$\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} = \left\{ \rho \in \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \;\middle|\; \begin{array}{c} I_2 \text{ and } \rho(\alpha) \text{ generate} \\ \text{a unipotent mold over } \mathbb{F}_2 \end{array} \right\}.$$

Denote by $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}$ the universal representation with unipotent mold over $\mathbb{F}_2$ on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. There exists no eigenvalue of $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)$ on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ in general, and hence we will construct a faithfully flat finite morphism $p_\alpha : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ such that there exist eigenvalues of $p_\alpha^*(\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha))$ on $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$.

Definition 7.1. Let us define a quasi-coherent sheaf $\mathcal{A}_\alpha$ of $\mathcal{O}_{\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}}$-algebras on $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ by

$$\mathcal{A}_\alpha := \mathcal{O}_{\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}}[X_\alpha]/(X_\alpha^2 - \det \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)).$$

Then set $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} := \mathcal{S}pec\mathcal{A}_\alpha$.

Remark 7.2. The canonical morphism $p_\alpha : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ is faithfully flat and finite. Since $\mathrm{tr}(\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)) = 0$, X_α is an eigenvalue of $p_\alpha^*(\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha))$ on $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$. We see that

$$\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} = \left\{ (\rho, X) \;\middle|\; \rho \in \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \text{ and } X^2 = \det \rho(\alpha) \right\}.$$

For simplicity, we put $\widetilde{R}_\alpha := \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ and $R_\alpha := \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. Remark that $\mathcal{O}_{R_\alpha}[\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\Gamma)] = \mathcal{O}_{R_\alpha} \cdot I_2 + \mathcal{O}_{R_\alpha} \cdot \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)$ is a unipotent mold over $\mathbb{F}_2$ on R_α. For each $\gamma \in \Gamma$, we can write $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma) = a_\alpha(\gamma)I_2 + b_\alpha(\gamma)\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)$. Note that $a_\alpha(e) = 1, b_\alpha(e) = 0$ and that $a_\alpha(\alpha) = 0, b_\alpha(\alpha) = 1$.

Definition 7.3. For each γ, we define $r_\alpha(\gamma) \in \mathcal{O}_{\widetilde{R}_\alpha}(\widetilde{R}_\alpha)$ by $r_\alpha(\gamma) = a_\alpha(\gamma) + b_\alpha(\gamma)X_\alpha$. (In the sequel, we will omit p_α^*.)

Proposition 7.4. *For each $\gamma \in \Gamma$, $r_\alpha(\gamma)$ is an eigenvalue of $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)$ on $\widetilde{R}_\alpha$. In other words, $r_\alpha(\gamma)$ is a root of the characteristic polynomial of $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)$.*

Proof. By using $X_\alpha^2 = \det \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)$ and Lemma 4.10, we have

$$\begin{aligned} r_\alpha(\gamma)^2 &= (a_\alpha(\gamma) + b_\alpha(\gamma)X_\alpha)^2 \\ &= a_\alpha(\gamma)^2 + b_\alpha(\gamma)^2 X_\alpha^2 \\ &= a_\alpha(\gamma)^2 + b_\alpha(\gamma)^2 \det \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha) = \det \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma). \end{aligned}$$

Since $\mathrm{tr}(\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)) = 0$, the characteristic polynomial of $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)$ is $x^2 - \det \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)$. Hence $r_\alpha(\gamma)$ is an eigenvalue of $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)$. $\square$

Proposition 7.5. *For each $\alpha \in \Gamma$, $r_\alpha : \Gamma \to \mathcal{O}_{\widetilde{R}_\alpha}(\widetilde{R}_\alpha)$ is a character.*

Proof. Note that $a_\alpha(\gamma)$ and $b_\alpha(\gamma)$ are the (a,b)-coefficients of $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)$ with respect to $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)$. By Lemma 6.2, $r_\alpha(e) = a_\alpha(e) + b_\alpha(e)X_\alpha = 1$ and

$$\begin{aligned} r_\alpha(\gamma)r_\alpha(\delta) &= (a_\alpha(\gamma) + b_\alpha(\gamma)X_\alpha)(a_\alpha(\delta) + b_\alpha(\delta)X_\alpha) \\ &= (a_\alpha(\gamma)a_\alpha(\delta) + b_\alpha(\gamma)b_\alpha(\delta)X_\alpha^2) + (a_\alpha(\gamma)b_\alpha(\delta) + b_\alpha(\gamma)a_\alpha(\delta))X_\alpha \\ &= a_\alpha(\gamma\delta) + b_\alpha(\gamma\delta)X_\alpha = r_\alpha(\gamma\delta) \end{aligned}$$

for $\gamma, \delta \in \Gamma$. Here we used $X_\alpha^2 = \det \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)$. Hence r_α is a character of Γ. $\square$

For $\alpha, \beta \in \Gamma$, let us consider the pull-backs of the intersection $R_\alpha \cap R_\beta \subseteq \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ by $p_\alpha : \widetilde{R}_\alpha \to R_\alpha$ and $p_\beta : \widetilde{R}_\beta \to R_\beta$. Set $\widetilde{R}_{\alpha\beta} = p_\alpha^{-1}(R_\alpha \cap R_\beta)$ and $\widetilde{R}_{\beta\alpha} = p_\beta^{-1}(R_\beta \cap R_\alpha)$. We define the morphism $\phi_{\beta\alpha} : \widetilde{R}_{\alpha\beta} \to \widetilde{R}_{\beta\alpha}$ over $R_\alpha \cap R_\beta$ by $X_\beta \mapsto a_\alpha(\beta) + b_\alpha(\beta)X_\alpha$. Since $(a_\alpha(\beta) + b_\alpha(\beta)X_\alpha)^2 = a_\alpha(\beta)^2 + b_\alpha(\beta)^2 X_\alpha^2 = \det \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\beta)$ (see the proof of Proposition 7.4), we can define the morphism $\phi_{\beta\alpha}$.

It is easy to check that $\phi_{\alpha\alpha} = 1$, $\phi_{\alpha\beta} \circ \phi_{\beta\alpha} = 1$ and $\phi_{\gamma\beta} \circ \phi_{\beta\alpha} = \phi_{\gamma\alpha}$ over $R_\alpha \cap R_\beta \cap R_\gamma$ for $\alpha, \beta, \gamma \in \Gamma$. Gluing $\{\widetilde{R}_\alpha\}_{\alpha\in\Gamma}$, we have a scheme $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ over $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ by [6, Chap. II, Ex. 2.12]. The canonical morphism $p : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ is a faithfully flat finite morphism.

Let us define a $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-action on $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$. First, we define the action σ_α of $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ on $\widetilde{R}_\alpha$ as follows: For a Z-valued point $((\rho, X), P)$ of $\widetilde{R}_\alpha \times \mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ with an $\mathbb{F}_2$-scheme Z, we set $\sigma_\alpha((\rho, X), P) := (P^{-1}\rho P, X)$ as a Z-valued point of $\widetilde{R}_\alpha$. Since $X^2 = \det(\rho(\alpha)) = \det(P^{-1}\rho(\alpha)P)$, the morphism $\sigma_\alpha : \widetilde{R}_\alpha \times \mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2 \to \widetilde{R}_\alpha$ can be defined. It is easy to see that σ_α gives a group action.

Next, let us glue the actions $\{\sigma_\alpha\}_{\alpha\in\Gamma}$ of $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$. Recall that $\phi_{\beta\alpha} : \widetilde{R}_{\alpha\beta} \to \widetilde{R}_{\beta\alpha}$ over $R_\alpha \cap R_\beta$ is given by $X_\beta \mapsto a_\alpha(\beta) + b_\alpha(\beta)X_\alpha$. The proof of Proposition 6.7 shows that $a_\alpha(\beta)$ and $b_\alpha(\beta)$ are $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-invariant on R_α. Thereby, the actions σ_α and σ_β are compatible over $R_\alpha \cap R_\beta$, and hence we obtain the action σ of $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ on $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ by gluing $\{\sigma_\alpha\}_{\alpha\in\Gamma}$. Finally, remark that the canonical morphism $p : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ is $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-equivariant.

Let $A_1(\Gamma)$ be the coordinate ring of the affine scheme $\mathrm{Rep}_1(\Gamma)$. Set $A_1(\Gamma)_{\mathbb{F}_2} = A_1(\Gamma) \otimes_{\mathbb{Z}} \mathbb{F}_2$ and $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2} = \mathrm{Spec} A_1(\Gamma)_{\mathbb{F}_2}$. Let $\chi_\Gamma : \Gamma \to A_1(\Gamma)_{\mathbb{F}_2}$ be the universal character of Γ over $\mathbb{F}_2$.

Definition 7.6. For a $A_1(\Gamma)_{\mathbb{F}_2}$-module M, we define

$$\mathrm{Der}(\Gamma, M) = \left\{ \delta : \Gamma \to M \;\middle|\; \begin{array}{c} \delta(\alpha\beta) = \chi_\Gamma(\alpha)\delta(\beta) + \delta(\alpha)\chi_\Gamma(\beta) \\ \text{for } \alpha, \beta \in \Gamma \end{array} \right\}.$$

We can prove the following lemma in the same way as Lemma 5.11.

Lemma 7.7. *There exists a universal $A_1(\Gamma)_{\mathbb{F}_2}$-module $\Omega_{\Gamma/\mathbb{F}_2}$ representing the covariant functor*

$$\begin{array}{cccc} \mathrm{Der}(\Gamma, -) & : & (A_1(\Gamma)_{\mathbb{F}_2}\text{-}\mathbf{Mod}) & \to & (A_1(\Gamma)_{\mathbb{F}_2}\text{-}\mathbf{Mod}) \\ & & M & \mapsto & \mathrm{Der}(\Gamma, M). \end{array}$$

In particular,

$$\mathrm{Der}(\Gamma, M) \xrightarrow{\sim} \mathrm{Hom}_{A_1(\Gamma)_{\mathbb{F}_2}}(\Omega_{\Gamma/\mathbb{F}_2}, M)$$

is an isomorphism for each $A_1(\Gamma)_{\mathbb{F}_2}$-module M.

Remark 7.8. Let $d : \Gamma \to \Omega_{\Gamma/\mathbb{F}_2}$ be the universal derivation of Γ. We see that $\Omega_{\Gamma/\mathbb{F}_2}$ is generated by $\{d\gamma \mid \gamma \in \Gamma\}$ as an $A_1(\Gamma)_{\mathbb{F}_2}$-module. As in Remark 5.12, we see that if Γ is finitely generated, then $\Omega_{\Gamma/\mathbb{F}_2}$ is a finitely generated $A_1(\Gamma)_{\mathbb{F}_2}$-module.

Definition 7.9. We define the scheme $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ over $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ by

$$\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2} = \mathrm{Proj}\, S(\Omega_{\Gamma/\mathbb{F}_2}),$$

where $S(\Omega_{\Gamma/\mathbb{F}_2})$ is the symmetric algebra of $\Omega_{\Gamma/\mathbb{F}_2}$ over $A_1(\Gamma)_{\mathbb{F}_2}$.

Example 7.10 (*cf.* Example 5.14)**.** Let $\Upsilon_1 = \langle \alpha_0 \rangle$ be the free monoid of rank 1. The $A_1(\Upsilon_1)_{\mathbb{F}_2}$-module $\Omega_{\Upsilon_1/\mathbb{F}_2}$ is isomorphic to $A_1(\Upsilon_1)_{\mathbb{F}_2}$ by

$$\begin{array}{ccc} A_1(\Upsilon_1)_{\mathbb{F}_2} & \to & \Omega_{\Upsilon_1/\mathbb{F}_2} \\ 1 & \mapsto & d\alpha_0. \end{array}$$

In particular, $\widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2} \cong \mathrm{Rep}_1(\Upsilon_1)_{\mathbb{F}_2}$.

Let $\psi : X \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ be an $\mathbb{F}_2$-morphism. Let us regard $\Omega_{\Gamma/\mathbb{F}_2}$ as a quasi-coherent sheaf on $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$. There exists a one-to-one correspondence

$$\begin{aligned} &\mathrm{Hom}_{\,\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}}(X, \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}) \cong \\ &\qquad \{\psi^*(\Omega_{\Gamma/\mathbb{F}_2}) \twoheadrightarrow \mathcal{L} \to 0 \mid \mathcal{L} \text{ is a line bundle on } X\}/\sim. \end{aligned}$$

Here we say that $\psi^*(\Omega_{\Gamma/\mathbb{F}_2}) \overset{f_1}{\twoheadrightarrow} \mathcal{L}_1$ and $\psi^*(\Omega_{\Gamma/\mathbb{F}_2}) \overset{f_2}{\twoheadrightarrow} \mathcal{L}_2$ are equivalent if there exists an isomorphism $g : \mathcal{L}_1 \overset{\cong}{\to} \mathcal{L}_2$ such that $g \circ f_1 = f_2$.

For $\alpha \in \Gamma$, we define the open subscheme $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ of $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ by $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} := D(d\alpha) = \{d\alpha \neq 0\}$. For an $\mathbb{F}_2$-morphism $\psi : X \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$, there exists a one-to-one correspondence

$$\begin{aligned} &\mathrm{Hom}_{\,\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}}(X, \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}) \cong \\ &\left\{ \psi^*(\Omega_{\Gamma/\mathbb{F}_2}) \twoheadrightarrow \mathcal{L} \to 0 \,\middle|\, \begin{array}{l} \mathcal{L} \text{ is a line bundle on } X \text{ and } \psi^*(d\alpha) \text{ is} \\ \text{nowhere vanishing as a section of } \mathcal{L} \end{array} \right\} \Big/ \sim. \end{aligned}$$

When $\mathcal{L}$ is generated by $\psi^*(d\alpha)$, $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$. Let $r : \Gamma \to \Gamma(X, \mathcal{O}_X)$ be the character associated to $\psi : X \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$. Regarding $\psi^*(d\alpha)$ as 1 of $\mathcal{O}_X$, we have the following:

$$\begin{aligned} &\mathrm{Hom}_{\,\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}}(X, \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}) \cong \\ &\qquad \left\{ d \,\middle|\, \begin{array}{l} d : \Gamma \to \Gamma(X, \mathcal{O}_X) \text{ is a derivation with respect to } r \\ \text{such that } d(\alpha) = 1 \end{array} \right\}, \end{aligned}$$

where we say that $d : \Gamma \to \Gamma(X, \mathcal{O}_X)$ is a *derivation with respect to* r if $d(\gamma\delta) = r(\gamma)d(\delta) + d(\gamma)r(\delta)$ holds for each $\gamma, \delta \in \Gamma$.

We construct a morphism $\lambda : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$. By Proposition 7.5, $r_\alpha : \Gamma \to \mathcal{O}_{\widetilde{R}_\alpha}(\widetilde{R}_\alpha)$ is a character for each $\alpha \in \Gamma$. It gives us a morphism $\lambda_\alpha : \widetilde{R}_\alpha \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$. Through the isomorphism $\phi_{\beta\alpha} : \widetilde{R}_{\alpha\beta} \to \widetilde{R}_{\beta\alpha}$, λ_α and λ_β coincide on $\widetilde{R}_\alpha \cap \widetilde{R}_\beta$ for each $\alpha, \beta \in \Gamma$. Indeed, $\phi_{\beta\alpha}$ is given by $X_\beta \mapsto a_\alpha(\beta) + b_\alpha(\beta)X_\alpha$. By comparing

$$\begin{aligned}\sigma_{\Gamma,u/\mathbb{F}_2,\beta}(\gamma) &= a_\beta(\gamma)I_2 + b_\beta(\gamma)\sigma_{\Gamma,u/\mathbb{F}_2,\beta}(\beta)\\ &= a_\beta(\gamma)I_2 + b_\beta(\gamma)(a_\alpha(\beta)I_2 + b_\alpha(\beta)\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha))\\ &= (a_\beta(\gamma) + b_\beta(\gamma)a_\alpha(\beta))I_2 + b_\beta(\gamma)b_\alpha(\beta)\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)\end{aligned}$$

with

$$\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma) = a_\alpha(\gamma)I_2 + b_\alpha(\gamma)\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha),$$

we have $a_\alpha(\gamma) = a_\beta(\gamma) + b_\beta(\gamma)a_\alpha(\beta)$ and $b_\alpha(\gamma) = b_\beta(\gamma)b_\alpha(\beta)$ on $R_\alpha \cap R_\beta$ for each $\gamma \in \Gamma$. The isomorphism $\phi_{\beta\alpha}$ induces $\lambda_\beta(\gamma) = a_\beta(\gamma) + b_\beta(\gamma)X_\beta \mapsto a_\beta(\gamma) + b_\beta(\gamma)(a_\alpha(\beta) + b_\alpha(\beta)X_\alpha) = a_\alpha(\gamma) + b_\alpha(\gamma)X_\alpha = \lambda_\alpha(\gamma)$. Hence λ_α and λ_β coincide on $\widetilde{R}_\alpha \cap \widetilde{R}_\beta$. By gluing $\{\lambda_\alpha\}_{\alpha\in\Gamma}$, we obtain a morphism $\lambda : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$. We regard $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ as a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-scheme by λ.

We construct a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-morphism $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$. First, let us define $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2,\alpha} : \widetilde{R}_\alpha = \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ for each $\alpha \in \Gamma$. Put $\widetilde{C}_\alpha = \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ and $\widetilde{\pi}_\alpha := \widetilde{\pi}_{\Gamma,u/\mathbb{F}_2,\alpha}$ for simplicity. Set $\eta_\alpha(\gamma) := \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma) - r_\alpha(\gamma)I_2 \in \mathrm{M}_2(\mathcal{O}_{\widetilde{R}_\alpha}(\mathcal{O}_{\widetilde{R}_\alpha}))$ for $\gamma \in \Gamma$. By Proposition 7.4, $\eta_\alpha(\gamma)^2 = \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)^2 + r_\alpha(\gamma)^2 I_2 = \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)^2 + \det(\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma))I_2 = 0$. Note that $\mathcal{O}_{\widetilde{R}_\alpha}[\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\Gamma)] = \mathcal{O}_{\widetilde{R}_\alpha} \cdot I_2 + \mathcal{O}_{\widetilde{R}_\alpha} \cdot \sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha) = \mathcal{O}_{\widetilde{R}_\alpha} \cdot I_2 + \mathcal{O}_{\widetilde{R}_\alpha} \cdot \eta_\alpha(\alpha)$. For each $\gamma \in \Gamma$,

$$\begin{aligned}\eta_\alpha(\gamma) &= (a_\alpha(\gamma) - r_\alpha(\gamma))I_2 + b_\alpha(\gamma)\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)\\ &= -b_\alpha(\gamma)X_\alpha I_2 + b_\alpha(\gamma)\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\alpha)\\ &= b_\alpha(\gamma)\eta_\alpha(\alpha),\end{aligned}$$

since $r_\alpha(\alpha) = X_\alpha$.

Proposition 7.11. *For each $\alpha \in \Gamma$, $b_\alpha(\cdot) : \Gamma \to \mathcal{O}_{\widetilde{R}_\alpha}(\widetilde{R}_\alpha)$ is a derivation with respect to r_α.*

Proof. By calculating $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma\delta)$ and $\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\gamma)\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\delta)$, we have $b_\alpha(\gamma\delta) = a_\alpha(\gamma)b_\alpha(\delta) + b_\alpha(\gamma)a_\alpha(\delta)$ for each $\gamma, \delta \in \Gamma$. It follows that $b_\alpha(\gamma\delta) = r_\alpha(\gamma)b_\alpha(\delta) + b_\alpha(\gamma)r_\alpha(\delta)$. □

Hence we have a morphism $\widetilde{\pi}_\alpha : \widetilde{R}_\alpha \to \widetilde{C}_\alpha$ by the derivation $b_\alpha(\cdot) : \Gamma \to \mathcal{O}_{\widetilde{R}_\alpha}(\widetilde{R}_\alpha)$ with $b_\alpha(\alpha) = 1$.

Next, let us glue the morphisms $\{\widetilde{\pi}_\alpha : \widetilde{R}_\alpha \to \widetilde{C}_\alpha\}_{\alpha\in\Gamma}$. Because $b_\alpha(\gamma) = b_\beta(\gamma)b_\alpha(\beta)$ on $R_\alpha \cap R_\beta$ for each $\gamma \in \Gamma$ and $b_\alpha(\beta) \in (\mathcal{O}_{R_\alpha\cap R_\beta})^\times$, we have the following commutative diagram:

$$\begin{array}{ccccc} \lambda_\alpha^*(\Omega_{u/\mathbb{F}_2})\,|_{\widetilde{R}_\alpha\cap\widetilde{R}_\beta} & \twoheadrightarrow & \mathcal{O}_{\widetilde{R}_\alpha\cap\widetilde{R}_\beta} & \to 0 \\ || & & \uparrow_{b_\alpha(\beta)\cdot} & \\ \lambda_\beta^*(\Omega_{u/\mathbb{F}_2})\,|_{\widetilde{R}_\alpha\cap\widetilde{R}_\beta} & \twoheadrightarrow & \mathcal{O}_{\widetilde{R}_\alpha\cap\widetilde{R}_\beta} & \to 0, \end{array}$$

where the morphism $b_\alpha(\beta)\cdot : \mathcal{O}_{\widetilde{R}_\alpha\cap\widetilde{R}_\beta} \to \mathcal{O}_{\widetilde{R}_\alpha\cap\widetilde{R}_\beta}$ defined by $\varphi \mapsto b_\alpha(\beta)\varphi$ is an isomorphism. It follows that $\widetilde{\pi}_\alpha\,|_{\widetilde{R}_\alpha\cap\widetilde{R}_\beta} = \widetilde{\pi}_\beta\,|_{\widetilde{R}_\alpha\cap\widetilde{R}_\beta}$ for each $\alpha, \beta \in \Gamma$. Therefore we have a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-morphism $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2}$: $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$.

Let $\mathrm{PGL}_2 \otimes_\mathbb{Z} \mathbb{F}_2$ act on $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ trivially. Then we have the following proposition:

Proposition 7.12. *The morphism* $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ *is* $\mathrm{PGL}_2 \otimes_\mathbb{Z} \mathbb{F}_2$*-equivariant.*

Proof. It suffices to show that $\widetilde{\pi}_\alpha : \widetilde{R}_\alpha \to \widetilde{C}_\alpha$ is $\mathrm{PGL}_2 \otimes_\mathbb{Z} \mathbb{F}_2$-equivariant for each $\alpha \in \Gamma$. The character r_α on $\widetilde{R}_\alpha$ is given by $r_\alpha(\gamma) = a_\alpha(\gamma) + b_\alpha(\gamma)X_\alpha$ for $\gamma \in \Gamma$. The proof of Proposition 6.7 shows that $a_\alpha(\gamma)$ and $b_\alpha(\gamma)$ are $\mathrm{PGL}_2 \otimes_\mathbb{Z} \mathbb{F}_2$-invariant on R_α. From the definition of the action σ_α on $\widetilde{R}_\alpha$, X_α is $\mathrm{PGL}_2 \otimes_\mathbb{Z} \mathbb{F}_2$-invariant. Hence $r_\alpha(\gamma)$ is also $\mathrm{PGL}_2 \otimes_\mathbb{Z} \mathbb{F}_2$-invariant for each $\gamma \in \Gamma$. The morphism $\widetilde{\pi}_\alpha$ is given by the derivation $b_\alpha(\cdot)$ with respect to r_α. Since $b_\alpha(\gamma)$ is $\mathrm{PGL}_2 \otimes_\mathbb{Z} \mathbb{F}_2$-invariant, the morphism $\widetilde{\pi}_\alpha$ is $\mathrm{PGL}_2 \otimes_\mathbb{Z} \mathbb{F}_2$-equivariant. This completes the proof. □

Let us define a morphism $q_\alpha : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ for each $\alpha \in \Gamma$. By the definitions, $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ and $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ are $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-schemes. Let r and d be the universal characters on $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ and $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$, respectively. Consider the character r^2 on $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ instead of r. For constructing q_α, it suffices to define (a, b)-coefficients with respect to (r^2, α) on $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$. Denote by δ the universal derivation with respect to r on $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ such that $\delta(\alpha) = 1$. Then we define $a(\gamma) = r(\gamma) - r(\alpha)\delta(\gamma)$ and $b(\gamma) = \delta(\gamma)$ for $\gamma \in \Gamma$.

Proposition 7.13. *Let a and b be as above. Then a and b are (a, b)-coefficients with respect to (r^2, α) on* $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$.

Proof. It is easy to check that $a(e) = 1, b(e) = 0$ and that $a(\alpha) = 0, b(\alpha) = 1$. By direct calculations, we have

$$\begin{aligned} a(\gamma_1\gamma_2) &= r(\gamma_1\gamma_2) - r(\alpha)\delta(\gamma_1\gamma_2) \\ &= r(\gamma_1)r(\gamma_2) - r(\alpha)r(\gamma_1)\delta(\gamma_2) - r(\alpha)\delta(\gamma_1)r(\gamma_2) \\ &= (r(\gamma_1) - r(\alpha)\delta(\gamma_1))(r(\gamma_2) - r(\alpha)\delta(\gamma_2)) + r(\alpha)^2\delta(\gamma_1)\delta(\gamma_2) \\ &= a(\gamma_1)a(\gamma_2) + r^2(\alpha)b(\gamma_1)b(\gamma_2), \end{aligned}$$

$$\begin{aligned} b(\gamma_1\gamma_2) &= \delta(\gamma_1\gamma_2) \\ &= r(\gamma_1)\delta(\gamma_2) + \delta(\gamma_1)r(\gamma_2) \\ &= (r(\gamma_1) - r(\alpha)\delta(\gamma_1))\delta(\gamma_2) + \delta(\gamma_1)(r(\gamma_2) - r(\alpha)\delta(\gamma_2)) \\ &= a(\gamma_1)b(\gamma_2) + b(\gamma_1)a(\gamma_2), \end{aligned}$$

and

$$\begin{aligned} a(\gamma)^2 + b(\gamma)^2r^2(\alpha) &= (a(\gamma) + b(\gamma)r(\alpha))^2 \\ &= (r(\gamma) - r(\alpha)\delta(\gamma) + \delta(\gamma)r(\alpha))^2 \\ &= r^2(\gamma). \end{aligned}$$

Hence we have proved the statement. $\square$

By the (a, b)-coefficients on $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$, we obtain a morphism $q_\alpha : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. Denote by $r^2 : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ and $d : \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ the morphisms induced by the characters r^2 and d, respectively. Then $d \circ q_\alpha = r^2$.

Thus, we have the following commutative diagram for each $\alpha \in \Gamma$:

$$\begin{array}{ccc} \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} & \xrightarrow{p_\alpha} & \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \\ \widetilde{\pi}_\alpha \downarrow & & \downarrow \pi_\alpha \\ \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} & \xrightarrow{q_\alpha} & \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}, \end{array}$$

where we denote by π_α the morphism $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ which was defined in §6. These morphisms are $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-equivariant. Let $\lambda_\alpha^2 : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ and $\det : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ be the morphisms corresponding to the characters r_α^2 and $\det(\sigma_{\Gamma,u/\mathbb{F}_2,\alpha}(\cdot))$, respectively. Then $\det \circ p_\alpha = \lambda_\alpha^2$.

Proposition 7.14. *The commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} & \xrightarrow{p_\alpha} & \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \\ \widetilde{\pi}_\alpha \downarrow & & \downarrow \pi_\alpha \\ \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} & \xrightarrow{q_\alpha} & \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}. \end{array}$$

gives a fibre product for each $\alpha \in \Gamma$.

Proof. Put $R_\alpha := \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ and $C_\alpha := \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. We show that $(p_\alpha, \widetilde{\pi}_\alpha) : \widetilde{R}_\alpha \to R_\alpha \times_{C_\alpha} \widetilde{C}_\alpha$ is an isomorphism. It suffices to prove that $(p_\alpha, \widetilde{\pi}_\alpha)$ induces a bijective map between the sets of Z-valued points for any $\mathbb{F}_2$-scheme Z. Let (ρ_1, X_1) and (ρ_2, X_2) be Z-valued points of $\widetilde{R}_\alpha$ such that the images by $(p_\alpha, \widetilde{\pi}_\alpha)$ coincide. Obviously, $\rho_1 = \rho_2$. By the assumption, (ρ_1, X_1) and (ρ_2, X_2) induce the same character r on $\widetilde{R}_\alpha$. Since $X_1 = r(\alpha) = X_2$ on Z, $(\rho_1, X_1) = (\rho_2, X_2)$. Hence we have proved the injectivity.

Let $(\rho, (r, \delta))$ be a Z-valued point of $R_\alpha \times_{C_\alpha} \widetilde{C}_\alpha$, where $r : \Gamma \to \mathcal{O}_Z(Z)$ is a character and $\delta : \Gamma \to \mathcal{O}_Z(Z)$ is a derivation with respect to r such that $\delta(\alpha) = 1$. The Z-valued point (a, b) of C_α induced by (r, δ) is given by $a(\gamma) = r(\gamma) - r(\alpha)\delta(\gamma)$ and $b(\gamma) = \delta(\gamma)$ for each $\gamma \in \Gamma$. Note that a and b are (a, b)-coefficients with respect to (r^2, α) by Proposition 7.13. Because ρ and (r, δ) induce the same Z-valued point of C_α, $r^2(\gamma) = \det \rho(\gamma)$ and $\rho(\gamma) = a(\gamma)I_2 + b(\gamma)\rho(\alpha) = (r(\gamma) - r(\alpha)\delta(\gamma))I_2 + \delta(\gamma)\rho(\alpha)$ for each $\gamma \in \Gamma$. Set $X := r(\alpha)$. Then (ρ, X) is a Z-valued point of $\widetilde{R}_\alpha$ by $X^2 = r^2(\alpha) = \det \rho(\alpha)$. It is obvious that $p_\alpha(\rho, X) = \rho$. Denote by (r', δ') the image of (ρ, X) by $\widetilde{\pi}_\alpha$. For each $\gamma \in \Gamma$, $\delta'(\gamma) = b(\gamma) = \delta(\gamma)$ and

$$
\begin{aligned}
r'(\gamma) &= a(\gamma) + b(\gamma)X \\
&= (r(\gamma) - r(\alpha)\delta(\gamma)) + \delta(\gamma)X \\
&= r(\gamma).
\end{aligned}
$$

Thus $\widetilde{\pi}_\alpha(\rho, X) = (r, \delta)$. Hence $(p_\alpha, \widetilde{\pi}_\alpha)(\rho, X) = (\rho, (r, \delta))$, which implies the surjectivity. Therefore we have proved the statement. $\square$

Definition 7.15. Let $\Upsilon_1 = \langle \alpha_0 \rangle$ be the free monoid of rank 1. As in Definition 6.10, we call the morphism $\widetilde{\pi}_{\Upsilon_1, u/\mathbb{F}_2, \alpha_0} : \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} \to \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$ the *prototype* in the unipotent mold over $\mathbb{F}_2$ case. Remark that $\widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2} = \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$ and that $\widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2} = \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$.

Remark 7.16. Recall that $\pi_{\Upsilon_1, u/\mathbb{F}_2, \alpha_0} : \mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0} \to \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ is a universal geometric quotient by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ (Theorem 6.11). The prototype $\pi_{\Upsilon_1, u/\mathbb{F}_2, \alpha_0}$ is described by $\mathrm{Spec}(\mathbb{F}_2[a, b, c, d]/(a+d)) \supset D(b) \cup D(c) \to \mathrm{Spec}\mathbb{F}_2[D]$, where D is mapped to $ad - bc$. Then $\widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$ is isomorphic to $D(b) \cup D(c) \subset \mathrm{Spec}(\mathbb{F}_2[a, b, c, d][X]/(a + d, X^2 - ad +$

$bc)$). Since $\Omega_{\Upsilon_1,u/\mathbb{F}_2}$ is isomorphic to $A_1(\Upsilon_1)_{\mathbb{F}_2}$, $\widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$ is isomorphic to $\mathrm{Rep}_1(\Upsilon_1)_{\mathbb{F}_2} \cong \mathrm{Spec}\mathbb{F}_2[\chi]$. Here the indeterminate χ corresponds to the value at α_0 of the universal character on $\mathrm{Rep}_1(\Upsilon_1)_{\mathbb{F}_2}$. Therefore $\widetilde{\pi}_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0} : \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} \to \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$ is described by $\mathrm{Spec}(\mathbb{F}_2[a,b,c,d][X]/(a+d, X^2-ad+bc)) \supset D(b)\cup D(c) \to \mathrm{Spec}\mathbb{F}_2[\chi]$, where χ is mapped to X.

By Proposition 7.14, the prototype $\widetilde{\pi}_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0} : \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} \to \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$ is obtained by base change of $\pi_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0} : \mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0} \to \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$. Theorem 6.11 implies the following:

Theorem 7.17. *The prototype*

$$\widetilde{\pi}_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0} : \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} \to \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$$

is a universal geometric quotient by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$.

Let Γ be a group or a monoid. For $\alpha \in \Gamma$, we define the monoid homomorphism $\phi : \Upsilon_1 = \langle \alpha_0 \rangle \to \Gamma$ by $\alpha_0 \mapsto \alpha$. By restricting representations, characters, and derivations of Γ to those of Υ_1 through ϕ, we can obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} & \to & \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} & \to & \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}. \end{array}$$

Under this situation, we have the following lemma.

Lemma 7.18. *The above diagram gives a fibre product. In particular, the morphism* $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ *is obtained by base change of the prototype.*

Proof. Here we prove the statement without using Lemma 6.12. Put $\widetilde{R}_{\alpha_0} := \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$ and $\widetilde{C}_{\alpha_0} := \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$. It suffices to prove that $\widetilde{R}_\alpha \to \widetilde{R}_{\alpha_0} \times_{\widetilde{C}_{\alpha_0}} \widetilde{C}_\alpha$ induces a bijective map between the sets of Z-valued points for any $\mathbb{F}_2$-scheme Z. Let (ρ_1, X_1) and (ρ_2, X_2) be Z-valued points of $\widetilde{R}_\alpha$ whose images coincide. By the assumption, $X_1 = X_2$ and $\rho_1(\alpha) = \rho_1(\phi(\alpha_0)) = \rho_2(\phi(\alpha_0)) = \rho_2(\alpha)$. Since (ρ_1, X_1) and (ρ_2, X_2) induce the same Z-valued point (r, δ) of $\widetilde{C}_\alpha$, $a(\gamma) = r(\gamma) - r(\alpha)\delta(\gamma)$ and $b(\gamma) = \delta(\gamma)$ have the same values for (ρ_1, X_1) and (ρ_2, X_2). It follows that $\rho_1(\gamma) = a(\gamma)I_2 + b(\gamma)\rho_1(\alpha) = a(\gamma)I_2 + b(\gamma)\rho_2(\alpha) = \rho_2(\gamma)$ for each $\gamma \in \Gamma$. Hence $(\rho_1, X_1) = (\rho_2, X_2)$, which implies the injectivity.

Let $((\rho_0, X_0), (r, \delta))$ be a Z-valued point of $\widetilde{R}_{\alpha_0} \times_{\widetilde{C}_{\alpha_0}} \widetilde{C}_\alpha$, where $r : \Gamma \to \mathcal{O}_Z(Z)$ is a character and $\delta : \Gamma \to \mathcal{O}_Z(Z)$ is a derivation with respect to r such that $\delta(\alpha) = 1$. Put $a(\gamma) = r(\gamma) - r(\alpha)\delta(\gamma)$ and $b(\gamma) = \delta(\gamma)$ for each $\gamma \in \Gamma$. By Proposition 7.13, a and b are (a, b)-coefficients with respect to (r^2, α). Set $X := X_0$ and $\rho(\gamma) := a(\gamma)I_2 + b(\gamma)\rho_0(\alpha_0)$ for $\gamma \in \Gamma$. Since $a(\alpha) = 0$ and $b(\alpha) = 1$, $\rho(\alpha) = \rho_0(\alpha_0)$. Note that $r^2(\alpha) = r^2(\phi(\alpha_0)) = X_0^2 = \det \rho_0(\alpha_0)$. It follows from Lemma 6.3 that ρ is a representation with unipotent mold over $\mathbb{F}_2$ such that $\det \rho(\gamma) = r^2(\gamma)$ for each $\gamma \in \Gamma$. Then (ρ, X) is a Z-valued point of $\widetilde{R}_\alpha$. It is easy to check that (ρ, X) is mapped to $((\rho_0, X_0), (r, \delta))$. This implies the surjectivity. Hence we have proved the statement. $\square$

Theorem 7.19. *The morphism* $\widetilde{\pi}_{\Gamma, u/\mathbb{F}_2, \alpha} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2, \alpha} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2, \alpha}$ *is a universal geometric quotient by* $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ *for each* $\alpha \in \Gamma$. *Furthermore,* $\widetilde{\pi}_{\Gamma, u/\mathbb{F}_2} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ *is a universal geometric quotient by* $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$.

Proof. The statement follows from Theorem 7.17 and Lemma 7.18. $\square$

The following Lemma states the "descent" of universal geometric quotients. The proof was suggested by Michiaki Inaba.

Lemma 7.20. *Let G be a group scheme separated of finite type over a scheme S. Let $\phi : X \to Y$ be a G-equivariant separated morphism of finite type over S, where G acts on Y trivially. For a faithfully flat and quasi-compact morphism $f : Y' \to Y$, put $X' := X \times_Y Y'$ and $\phi' : X' \to Y'$. If ϕ' is a (universal) geometric quotient by G, then ϕ is also a (resp. universal) geometric quotient by G.*

Proof. It suffices to prove that if ϕ' is a geometric quotient, so is ϕ. It is easy to see that ϕ is surjective and that the image of $G \times X \to X \times_S X$ is equal to $X \times_Y X$. If ϕ' is (universally) submersive, then so is ϕ by [5, Lemma 15.7.11.1]. Let $\sigma : G \times_S X \to X$ and $\sigma' : G \times_S X' \to X'$ be the groups actions of G on X and X', respectively. Denote the second projections by $p_2 : G \times_S X \to X$ and $p_2' : G \times_S X' \to X'$. Put $\tau := \phi \circ \sigma = \phi \circ p_2$, $\tau' := \phi' \circ \sigma' = \phi' \circ p_2'$, and $f' : X' \to X$. For proving that $\phi_*(\mathcal{O}_X)^G = \mathcal{O}_Y$, we show that $0 \to \mathcal{O}_Y \to \phi_*(\mathcal{O}_X) \overset{\sigma_* - p_{2*}}{\to} \tau_*(\mathcal{O}_{G \times_S X})$ is exact. Taking the pullback by f, we have $0 \to f^*\mathcal{O}_Y \to f^*\phi_*(\mathcal{O}_X) \overset{f^*(\sigma_* - p_{2*})}{\to} f^*\tau_*(\mathcal{O}_{G \times_S X})$. By [3, Proposition 1.4.15], $f^*\phi_*(\mathcal{O}_X) \cong \phi'_* f'^*(\mathcal{O}_X) \cong \phi'_*(\mathcal{O}_{X'})$ and $f^*\tau_*(\mathcal{O}_{G \times_S X}) \cong \tau'_*(1_G \times f')^*(\mathcal{O}_{G \times_S X}) \cong \tau'_*(\mathcal{O}_{G \times_S X'})$. Then we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \to & f^*(\mathcal{O}_Y) & \to & f^*\phi_*(\mathcal{O}_X) & \stackrel{f^*(\sigma_*-p_{2*})}{\to} & f^*\tau_*(\mathcal{O}_{G\times_S X}) \\ & & || & & \downarrow\cong & & \downarrow\cong \\ 0 & \to & \mathcal{O}_{Y'} & \to & \phi'_*(\mathcal{O}'_X) & \stackrel{\sigma'_*-p'_{2*}}{\to} & \tau'_*(\mathcal{O}_{G\times_S X'}). \end{array}$$

Since ϕ' is a geometric quotient, the complex $0 \to \mathcal{O}_{Y'} \to \phi'_*(\mathcal{O}'_X) \stackrel{\sigma'_*-p'_{2*}}{\to} \tau'_*(\mathcal{O}_{G\times_S X'})$ is exact, and hence so is $0 \to f^*\mathcal{O}_Y \to f^*\phi_*(\mathcal{O}_X) \stackrel{f^*(\sigma_*-p_{2*})}{\to} f^*\tau_*(\mathcal{O}_{G\times_S X})$. Because f is faithfully flat, $0 \to \mathcal{O}_Y \to \phi_*(\mathcal{O}_X) \stackrel{\sigma_*-p_{2*}}{\to} \tau_*(\mathcal{O}_{G\times_S X})$ is also exact. Thus we have proved the statement. □

Now we can prove Theorem 6.13 by another approach.

Theorem 7.21 (Theorem 6.13)**.** *The morphism* $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$ *is a universal geometric quotient by* $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ *for each* $\alpha \in \Gamma$*.*

Proof. The statement follows from Proposition 7.14, Theorem 7.19, and Lemma 7.20. □

By the same discussion in §6, we can construct $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ by gluing $\{\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}\}_{\alpha\in\Gamma}$. Then we have Corollary 6.14, which states that $\pi_{\Gamma,u/\mathbb{F}_2} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is a universal geometric quotient by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$. Similarly, we have a morphism $q : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ by gluing $\{q_\alpha : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}\}_{\alpha\in\Gamma}$.

Remark 7.22. By Remark 7.16, we see that $q : \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} \to \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ is described as $\mathrm{Spec}\widetilde{\mathbb{F}_2[\chi]} \to \mathrm{Spec}\mathbb{F}_2[D]$, where D is mapped to χ^2. We also see that $p : \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} \to \mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}$ is described as $\mathrm{Spec}\ \mathbb{F}_2[a,b,c,X]/(X^2 + a^2 + bc) \supset D(b) \cup D(c) \to D(b) \cup D(c) \subset \mathrm{Spec}\mathbb{F}_2[a,b,c]$. Hence p and q are faithfully flat finite morphisms of finite presentation, but not smooth morphisms.

For $\alpha \in \Gamma$, the monoid homomorphism $\phi : \Upsilon_1 \to \Gamma$ by $\alpha_0 \mapsto \alpha$ induces the following commutative diagrams:

$$\begin{array}{ccc} \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} & \to & \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} & \to & \mathrm{Rep}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0} \end{array}$$

and

$$\begin{array}{ccc} \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} & \to & \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} & \to & \mathrm{Ch}_2(\Upsilon_1)_{u/\mathbb{F}_2,\alpha_0}. \end{array}$$

We can show that the diagrams above give fibre products in the same way as Lemma 7.18.

From the discussions above, we obtain the following commutative diagram which gives a fibre product:

$$\begin{array}{rcl} \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} & \xrightarrow{p} & \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \\ \widetilde{\pi}_{\Gamma,u/\mathbb{F}_2} \downarrow & & \downarrow \pi_{\Gamma,u/\mathbb{F}_2} \\ \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2} & \xrightarrow{q} & \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}. \end{array}$$

All morphisms are $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$-equivariant, and p and q are faithfully flat finite morphisms of finite presentation. Note that p and q are not smooth in general.

Recall that $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ can be regarded as a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-scheme by $\lambda : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$. Let r be the corresponding character on $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ to λ, and let $\lambda^2 : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ be the morphism induced by the character r^2. Denote by $\det : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ the morphism corresponding to the character $\det(\sigma_{\Gamma,u/\mathbb{F}_2}(\cdot)))$. Then $\det \circ p = \lambda^2$.

By Definition 7.9, $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ is a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-scheme. Let us denote by $r : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ the canonical morphism. We also denote by the same symbol r the corresponding character on $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ to r. We define $r^2 : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$ as the morphism induced by the character r^2 on $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$. By Remark 6.15, $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is also a $\mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$-scheme by $d : \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2} \to \mathrm{Rep}_1(\Gamma)_{\mathbb{F}_2}$. Then $d \circ q = r^2$.

Remark 7.23. The morphism $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ is smooth and surjective for each group or monoid Γ. Indeed, $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2,\alpha} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ is obtained by base change of the prototype by Lemma 7.18. The prototype $\widetilde{\pi}_{\Upsilon_1,u/\mathbb{F}_2,\alpha_0} : \widetilde{\mathrm{Rep}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0} \to \widetilde{\mathrm{Ch}_2(\Upsilon_1)}_{u/\mathbb{F}_2,\alpha_0}$ is smooth and surjective because it is obtained by base change of $\pi : \mathrm{Rep}_2(\Upsilon_1)_{\mathrm{rk}2} \to \mathrm{Ch}_2(\Upsilon_1)$ and π is smooth and surjective by Proposition 4.9.

Example 7.24. Let us describe $\widetilde{\mathrm{Ch}_2(\Upsilon_m)}_{u/\mathbb{F}_2}$ for the free monoid $\Upsilon_m = \langle \alpha_1, \dots, \alpha_m \rangle$ of rank m. Put $\widetilde{C(m)} := \widetilde{\mathrm{Ch}_2(\Upsilon_m)}_{u/\mathbb{F}_2}$. Let $A_1(m)_{\mathbb{F}_2}$ denote the coordinate ring $A_1(\Upsilon_m)_{\mathbb{F}_2}$ of $\mathrm{Rep}_1(\Upsilon_m)_{\mathbb{F}_2}$. We can write $A_1(m)_{\mathbb{F}_2} = \mathbb{F}_2[\chi(\alpha_1), \dots, \chi(\alpha_m)]$, where $\chi(\alpha_1), \dots, \chi(\alpha_m)$ are indeterminates. It is easy to see that the $A_1(m)_{\mathbb{F}_2}$-module $\Omega_{\Upsilon_m/\mathbb{F}_2}$ is isomorphic to the free module $\oplus_{i=1}^m A_1(m)_{\mathbb{F}_2} \cdot d\alpha_i$. Hence $\widetilde{C(m)} = \mathrm{Proj}\, S(\Omega_{\Upsilon_m/\mathbb{F}_2})$ is isomorphic to $\mathbb{A}^m_{\mathbb{F}_2} \times \mathbb{P}^{m-1}_{\mathbb{F}_2}$. The projection $\widetilde{C(m)} \to \mathrm{Rep}_1(\Upsilon_m)_{\mathbb{F}_2}$ can be described by the first projection $p_1 : \mathbb{A}^m_{\mathbb{F}_2} \times \mathbb{P}^{m-1}_{\mathbb{F}_2} \to \mathbb{A}^m_{\mathbb{F}_2} = \mathrm{Rep}_1(\Upsilon_m)_{\mathbb{F}_2}$.

Put $\widetilde{C(m)}_i := \{d\alpha_i \neq 0\} \subset \widetilde{C(m)}$ for $1 \leq i \leq m$. Note that $\widetilde{C(m)}_i \cong \mathbb{A}^m_{\mathbb{F}_2} \times \mathbb{A}^{m-1}_{\mathbb{F}_2} \subset \mathbb{A}^m_{\mathbb{F}_2} \times \mathbb{P}^{m-1}_{\mathbb{F}_2}$. In Example 6.21, we have described $C(m) = \mathrm{Ch}_2(\Upsilon_m)_{u/\mathbb{F}_2}$ and $C(m)_i = \mathrm{Ch}_2(\Upsilon_m)_{u/\mathbb{F}_2,\alpha_i}$ for $1 \leq i \leq m$. The morphism $q : \widetilde{C(m)} \to C(m)$ can be described as follows: Let $q_i : \widetilde{C(m)}_i \to C(m)_i$ be the restriction of q to $\widetilde{C(m)}_i$ for $1 \leq i \leq m$. For $(r, \delta) \in \widetilde{C(m)}_i$, $q_i(r, \delta) = (a, b) \in C(m)_i$ is given by $a(\gamma) = r(\gamma) - r(\alpha_i)\delta(\gamma)$ and $b(\gamma) = \delta(\gamma)$ for $\gamma \in \Upsilon_m$. Recall that the isomorphisms $\widetilde{C(m)}_i \cong \mathbb{A}^{2m-1}_{\mathbb{F}_2}$ and $C(m)_i \cong \mathbb{A}^{2m-1}_{\mathbb{F}_2}$ are given by

$$\begin{aligned}(r, \delta) \mapsto (r(\alpha_1), \dots, r(\alpha_m), \delta(\alpha_1)/\delta(\alpha_i), \dots, \delta(\alpha_{i-1})/\delta(\alpha_i),& \\ \delta(\alpha_{i+1})/\delta(\alpha_i), \dots, \delta(\alpha_m)/\delta(\alpha_i))&\end{aligned}$$

and

$$\begin{aligned}(a_i, b_i, d) \mapsto (a_i(\alpha_1), \dots, a_i(\alpha_{i-1}), a_i(\alpha_{i+1}), \dots, a_i(\alpha_m), b_i(\alpha_1), \dots,& \\ b_i(\alpha_{i-1}), b_i(\alpha_{i+1}), \dots, b_i(\alpha_m), d(\alpha_i)),&\end{aligned}$$

respectively. By these isomorphisms, $q_i : \mathbb{A}^{2m-1}_{\mathbb{F}_2} \to \mathbb{A}^{2m-1}_{\mathbb{F}_2}$ is described by

$$\begin{aligned}&q_i(r_1, \cdots, r_m, \overline{\delta}_1, \dots, \overline{\delta}_{i-1}, \overline{\delta}_{i+1}, \dots, \overline{\delta}_m) = \\ &\qquad (r_1 - r_i\overline{\delta}_1, \dots, r_{i-1} - r_i\overline{\delta}_{i-1}, r_{i+1} - r_i\overline{\delta}_{i+1}, \dots, r_m - r_i\overline{\delta}_m, \\ &\qquad\qquad \overline{\delta}_1, \dots, \overline{\delta}_{i-1}, \overline{\delta}_{i+1}, \dots, \overline{\delta}_m, r_i^2).\end{aligned}$$

Set $\widetilde{R(m)} := \widetilde{\mathrm{Rep}_2(\Upsilon_m)}_{\mathbb{F}_2}$ and $R(m) := \mathrm{Rep}_2(\Upsilon_m)_{\mathbb{F}_2}$. For $1 \leq i \leq m$, put $\widetilde{R(m)}_i := \widetilde{\mathrm{Rep}_2(\Upsilon_m)}_{\mathbb{F}_2,\alpha_i}$ and $R(m)_i := \mathrm{Rep}_2(\Upsilon_m)_{\mathbb{F}_2,\alpha_i}$. Let $p_i : \widetilde{R(m)}_i \to R(m)_i$ be the restriction of $p : \widetilde{R(m)} \to R(m)$ to $\widetilde{R(m)}_i$ for $1 \leq i \leq m$. We can describe $p_i : \widetilde{R(m)}_i = \{(A_1, \dots, A_m, X_i) \mid (A_1, \dots, A_m) \in R(m)_i \text{ and } X_i^2 = \det A_i\} \to R(m)_i = \{(A_1, \dots, A_m) \mid \langle A_1, \dots, A_m \rangle = \langle A_i \rangle \text{ is a unipotent mold over } \mathbb{F}_2\}$ by $(A_1, \dots, A_m, X_i)$

$\mapsto (A_1, \ldots, A_m)$, where $A_j := \rho(\alpha_j)$ for $1 \le j \le m$ and for each representation ρ. Let $\widetilde{\pi}_i$ and π_i denote the restrictions of $\widetilde{\pi}_{\Upsilon_{m,u/\mathbb{F}_2}} : \widetilde{R(m)} \to \widetilde{C(m)}$ and $\pi_{\Upsilon_{m,u/\mathbb{F}_2}} : R(m) \to C(m)$ to $\widetilde{R(m)}_i$ and $R(m)_i$, respectively. For $(A_1, \ldots, A_m) \in R(m)_i$, we can write $A_j = \overline{a}_{ij} I_2 + \overline{b}_j A_i$ for $1 \le j \le m$. Then $\widetilde{\pi}_i : \widetilde{R(m)}_i \to \widetilde{C(m)}_i$ is described by $(A_1, \ldots, A_m, X_i) \mapsto (\overline{a}_{i1} + \overline{b}_1 X_i, \ldots, \overline{a}_{im} + \overline{b}_m X_i, \overline{b}_1, \ldots, \overline{b}_{i-1}, \overline{b}_{i+1}, \ldots, \overline{b}_m)$ and $\pi_i : R(m)_i \to C(m)_i$ is described by $(A_1, \ldots, A_m) \mapsto (\overline{a}_{i1}, \ldots, \overline{a}_{i,i-1}, \overline{a}_{i,i+1}, \ldots, \overline{a}_{im}, \overline{b}_1, \ldots, \overline{b}_{i-1}, \overline{b}_{i+1}, \ldots, \overline{b}_m, \det A_i)$. Remark that

$$\begin{array}{ccc} \widetilde{R(m)}_i & \stackrel{p_i}{\to} & R(m)_i \\ \widetilde{\pi}_i \downarrow & & \downarrow \pi_i \\ \widetilde{C(m)}_i & \stackrel{q_i}{\to} & C(m)_i. \end{array}$$

gives a fibre product.

Definition 7.25. Let X be an $\mathbb{F}_2$-scheme. By a *tilde representation with unipotent mold over* $\mathbb{F}_2$ for Γ on X, we understand a pair (ρ, λ) of a representation ρ of with unipotent mold over $\mathbb{F}_2$ for Γ on X and a character $\lambda : \Gamma \to \mathcal{O}_X(X)$ satisfying the following conditions:

(i) $\det(\rho(\gamma)) = \lambda(\gamma)^2$ for each $\gamma \in \Gamma$.
(ii) $\{\rho(\gamma) - \lambda(\gamma) I_2 \mid \gamma \in \Gamma\}$ spans a sub-line bundle of $\mathcal{O}_X[\rho(\Gamma)]$.

Remark 7.26. Let (ρ, λ) be a tilde representation with unipotent mold over $\mathbb{F}_2$ for Γ on an $\mathbb{F}_2$-scheme X. For each point $x \in X$, choose $\alpha_x \in \Gamma$ and a neighbourhood U_x of x such that $\mathcal{O}_{U_x}[\rho(\Gamma)] = \mathcal{O}_{U_x} \cdot I_2 \oplus \mathcal{O}_{U_x} \cdot \rho(\alpha_x)$. The condition (ii) in Definition 7.25 means that for each $\gamma \in \Gamma$ there exists $c \in \mathcal{O}_{U_x}(U_x)$ such that $\rho(\gamma) - \lambda(\gamma) I_2 = c(\rho(\alpha_x) - \lambda(\alpha_x) I_2)$. Since $\rho(\gamma) = (\lambda(\gamma) - c\lambda(\alpha_x)) I_2 + c\rho(\alpha_x)$, the (a, b)-coefficients of $\rho(\gamma)$ with respect to $\rho(\alpha_x)$ are given by $a_{\alpha_x}(\gamma) = \lambda(\gamma) - c\lambda(\alpha_x)$ and $b_{\alpha_x}(\gamma) = c$. Then $\lambda(\gamma) = a_{\alpha_x}(\gamma) + b_{\alpha_x}(\gamma) \lambda(\alpha_x)$ for each $\gamma \in \Gamma$. Note that $(\rho|_{U_x}, \lambda(\alpha_x))$ gives a U_x-valued point of $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2, \alpha_x}$. Considering the definition of $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$, we see that we can obtain an X-valued point of $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ by gluing $\{(\rho|_{U_x}, \lambda(\alpha_x))\}_{x \in X}$.

By Remark 7.26, we have:

Proposition 7.27. *The following functor is representable by* $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$*:*

$$\begin{array}{rcccl} \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2} & : & (\mathbf{Sch}/\mathbb{F}_2)^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{c} \textit{tilde rep. with unipotent} \\ \textit{mold over } \mathbb{F}_2 \textit{ for } \Gamma \textit{ on } X \end{array} \right\}. \end{array}$$

Remark 7.28. The condition (ii) in Definition 7.25 is necessary for Proposition 7.27. Indeed, in the case of the free monoid $\Upsilon_2 = \langle \alpha, \beta \rangle$ of rank 2, let $\rho : \Upsilon_2 \to \mathrm{M}_2(k[\epsilon]/(\epsilon^2))$ be the representation defined by

$$\rho(\alpha) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \rho(\beta) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where k is a field. Let $\lambda : \Upsilon_2 \to k[\epsilon]/(\epsilon^2)$ be the character defined by $\lambda(\alpha) = \epsilon$ and $\lambda(\beta) = 1$. Then ρ is a representation with unipotent mold over $\mathbb{F}_2$ for Υ_2. For each $\gamma \in \Upsilon_2$, $\det \rho(\gamma) = \lambda(\gamma)^2$. However, the condition (ii) in Definition 7.25 fails. The (a, b)-coefficients of $\rho(\beta)$ with respect to $\rho(\alpha)$ is given by $a_\alpha(\beta) = 1$ and $b_\alpha(\beta) = 1$. The equality $\lambda(\beta) = a_\alpha(\beta) + b_\alpha(\beta)\lambda(\alpha)$ does not hold. Hence $(\rho, \lambda(\alpha)) \in \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}(k[\epsilon]/(\epsilon^2))$ and $(\rho, \lambda(\beta)) \in \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\beta}(k[\epsilon]/(\epsilon^2))$ induce different morphisms from $k[\epsilon]/(\epsilon^2)$ to $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$. This means that (ρ, λ) does not canonically induce a morphism to $\widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ without the condition (ii).

Remark 7.29. For each point $x \in \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$, there exists a local section $\widetilde{s_x} : V_x \to \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ on a neighbourhood V_x of x such that $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2} \circ \widetilde{s_x} = id_{V_x}$. Indeed, take $\alpha \in \Gamma$ such that $x \in \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$. By Proposition 7.14, $\widetilde{\pi_{\Gamma,u/\mathbb{F}_2,\alpha}} : \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha} \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ is obtained by base change of $\pi_{\Gamma,u/\mathbb{F}_2,\alpha} : \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2,\alpha} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2,\alpha}$. Remark 6.18 follows that $\pi_{\Gamma,u/\mathbb{F}_2,\alpha}$ has a section $s_{\Gamma,\alpha}$. Hence $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2,\alpha}$ has a section $\widetilde{s_{\Gamma,\alpha}}$. We can take $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2,\alpha}$ as a neighbourhood V_x of x. It is easy to see that $(\rho, \lambda) = \widetilde{s_{\Gamma,\alpha}}(r, \delta)$ is described by $\rho(\gamma) = a(\gamma)I_2 + b(\gamma)\begin{pmatrix} 0 & -r(\alpha)^2 \\ 1 & 0 \end{pmatrix}$ and $\lambda(\gamma) = r(\gamma)$ for $\gamma \in \Gamma$, where $a(\gamma) = r(\gamma) - r(\alpha)\delta(\gamma)$ and $b(\gamma) = \delta(\gamma)$.

Lemma 7.30. *Let* $(\rho_1, \lambda_1), (\rho_2, \lambda_2)$ *be tilde representations with unipotent mold over* $\mathbb{F}_2$ *for a group (or a monoid)* Γ *on a scheme* X *over* $\mathbb{F}_2$*. Let* $f_i : X \to \widetilde{\mathrm{Rep}_2(\Gamma)}_{u/\mathbb{F}_2}$ *be the morphism associated to* (ρ_i, λ_i) *for* $i = 1, 2$*. If* $\widetilde{\pi}_{\Gamma,u/\mathbb{F}_2} \circ f_1 = \widetilde{\pi}_{\Gamma,u/\mathbb{F}_2} \circ f_2 : X \to \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$*, then for* *each $x \in X$ there exists $P_x \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood V_x of x such that $P_x^{-1}\rho_1 P_x = \rho_2$ and $\lambda_1 = \lambda_2$ on V_x.*

Proof. In the same way as Lemma 6.19, we can prove the statement. $\square$

By a *generalized tilde representation with unipotent mold over* $\mathbb{F}_2$ for Γ on X, we understand triples $\{(U_i, \rho_i, \lambda_i)\}_{i\in I}$ of an open set U_i and a tilde representation (ρ_i, λ_i) with unipotent mold over $\mathbb{F}_2$ for Γ on U_i satisfying the following three conditions:

(i) $\cup_{i\in I} U_i = X$,

(ii) for each $x \in U_i \cap U_j$, there exists $P_x \in \mathrm{GL}_2(\Gamma(V_x, \mathcal{O}_X))$ on a neighbourhood $V_x \subseteq U_i \cap U_j$ of x such that $P_x^{-1}\rho_i P_x = \rho_j$ on V_x.

(iii) $\lambda_i = \lambda_j$ on $U_i \cap U_j$ for each i, j.

Generalized tilde representations $\{(U_i, \rho_i, \lambda_i)\}_{i\in I}$ and $\{(V_j, \sigma_j, \mu_j)\}_{j\in J}$ with unipotent mold over $\mathbb{F}_2$ are called *equivalent* if $\{(U_i, \rho_i, \lambda_i)\}_{i\in I} \cup \{(V_j, \sigma_j, \mu_j)\}_{j\in J}$ is a generalized tilde representation with unipotent mold over $\mathbb{F}_2$ again. Let us define the contravariant functor $\widetilde{\mathcal{E}q\,\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}$:

$$\begin{array}{cccc} \widetilde{\mathcal{E}q\,\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2} : & (\mathbf{Sch}/\mathbb{F}_2)^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ \begin{array}{c} \text{gen. tilde rep. with unip.} \\ \text{mold over } \mathbb{F}_2 \text{ for } \Gamma \text{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

Theorem 7.31. *The scheme $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ is a fine moduli scheme associated to the functor $\widetilde{\mathcal{E}q\,\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}$ for a group or a monoid Γ. In other words, $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ represents the functor $\widetilde{\mathcal{E}q\,\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}$. The moduli $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ is separated over $\mathbb{F}_2$; if Γ is a finitely generated group or monoid, then $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ is of finite type over $\mathbb{F}_2$.*

Proof. In the same way as Theorem 6.20, we can prove that $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ represents the functor $\widetilde{\mathcal{E}q\,\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}$ by using Lemma 7.30. It follows from Definition 7.9 that $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ is separated over $\mathbb{F}_2$. If Γ is finitely generated, then $\Omega_{\Gamma/\mathbb{F}_2}$ is a finitely generated module over $A_1(\Gamma)_{\mathbb{F}_2}$ by Remark 7.8. Hence $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ is of finite type over $\mathbb{F}_2$. $\square$

Remark 7.32. For an associative algebra A over a commutative ring R over $\mathbb{F}_2$, we can construct $\widetilde{\pi}_{A,u/\mathbb{F}_2} : \widetilde{\mathrm{Rep}_2(A)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(A)}_{u/\mathbb{F}_2}$ in the same way as group or monoid cases. Indeed, for $c \in A$, $\widetilde{\mathrm{Rep}_2(A)}_{u/\mathbb{F}_2,c}$

is defined as in Definition 7.1. By gluing $\{\widetilde{\mathrm{Rep}_2(A)}_{u/\mathbb{F}_2,c}\}_{c\in A}$, we have an R-scheme $\widetilde{\mathrm{Rep}_2(A)}_{u/\mathbb{F}_2}$ such that $p : \widetilde{\mathrm{Rep}_2(A)}_{u/\mathbb{F}_2} \to \mathrm{Rep}_2(A)_{u/\mathbb{F}_2}$ is a faithfully flat finite morphism. Let $A_1(A)$ be the coordinate ring of $\mathrm{Rep}_1(A)$. As in Remark 5.25, we can construct $A_1(A)$-module $\Omega_{A/R}$ such that $\mathrm{Der}(A, M) \cong \mathrm{Hom}_{A_1(A)}(\Omega_{A/R}, M)$ for any $A_1(A)$-module M. Set $\widetilde{\mathrm{Ch}_2(A)}_{u/\mathbb{F}_2} := \mathrm{Proj}S(\Omega_{A/R})$. Then we can construct a $\mathrm{Rep}_1(A)$-morphism $\widetilde{\pi}_{A,u/\mathbb{F}_2} : \widetilde{\mathrm{Rep}_2(A)}_{u/\mathbb{F}_2} \to \widetilde{\mathrm{Ch}_2(A)}_{u/\mathbb{F}_2}$, which is a universal geometric quotient by $\mathrm{PGL}_2 \otimes_{\mathbb{Z}} R$. By a *tilde representation with unipotent mold over* $\mathbb{F}_2$ for A on an R-scheme X, we understand a pair (ρ, λ) of a representation ρ of with unipotent mold over $\mathbb{F}_2$ for A on X and an R-homomorphism $\lambda : A \to \mathcal{O}_X(X)$ satisfying the following conditions:

(i) $\det(\rho(c)) = \lambda(c)^2$ for each $c \in A$.
(ii) $\{\rho(c) - \lambda(c)I_2 \mid c \in A\}$ spans a sub-line bundle of $\mathcal{O}_X[\rho(A)]$.

As in Proposition 7.27, we see that $\widetilde{\mathrm{Rep}_2(A)}_{u/\mathbb{F}_2}$ represents the following contravariant functor:

$$\begin{array}{ccc} (\mathbf{Sch}/R)^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \left\{ \begin{array}{c} \text{tilde rep. with unipotent} \\ \text{mold over } \mathbb{F}_2 \text{ for } A \text{ on } X \end{array} \right\}. \end{array}$$

In a similar way as group or monoid cases, we can define generalized tilde representations with unipotent mold over $\mathbb{F}_2$ for A on an R-scheme X. The contravariant functor $\widetilde{\mathcal{E}q\,\mathcal{U}_2(A)}_{\mathbb{F}_2}$ from the category of R-schemes to the category of sets is defined as

$$\begin{array}{ccccc} \widetilde{\mathcal{E}q\,\mathcal{U}_2(A)}_{\mathbb{F}_2} & : & (\mathbf{Sch}/R)^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \begin{array}{c} \text{gen. tilde rep. with unip.} \\ \text{mold over } \mathbb{F}_2 \text{ for } A \text{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

We can prove that $\widetilde{\mathrm{Ch}_2(A)}_{u/\mathbb{F}_2}$ is the fine moduli associated to $\widetilde{\mathcal{E}q\,\mathcal{U}_2(A)}_{\mathbb{F}_2}$ in the same way as Theorem 7.31. The moduli $\widetilde{\mathrm{Ch}_2(A)}_{u/\mathbb{F}_2}$ is separated over R; if A is a finitely generated algebra over R, then $\widetilde{\mathrm{Ch}_2(A)}_{u/\mathbb{F}_2}$ is of finite type over R.

Example 7.33. Let k be a field of characteristic 2. Let $K = k(\alpha)$ be a purely inseparable extension of k of degree 2 with $\beta = \alpha^2 \in k$. Regarding K as a k-vector space of dimension 2, we have a k-algebra homomorphism $\rho : K \to \mathrm{End}_k(K) \cong \mathrm{M}_2(k)$ by $c \mapsto (c' \mapsto cc')$, which is a representation with unipotent mold over $\mathbb{F}_2$. The matrix $\rho(\alpha) \in \mathrm{M}_2(k)$ has no eigenvalue in k, but has an eigenvalue after base change

$k \to K$. This example is an interesting case of representations with unipotent mold over $\mathbb{F}_2$.

The universal representation $\sigma_{K,u/\mathbb{F}_2}$ with unipotent mold over $\mathbb{F}_2$ on $\mathrm{Rep}_2(K)_{u/\mathbb{F}_2}$ is characterized by $\sigma_{K,u/\mathbb{F}_2}(\alpha)$ because it is a k-algebra homomorphism and $K = k(\alpha)$. Since $\mathrm{tr}(\sigma_{K,u/\mathbb{F}_2}(\alpha)) = 0$ and $\sigma_{K,u/\mathbb{F}_2}(\alpha)^2 = \beta I_2$, we can write $\mathrm{Rep}_2(K)_{u/\mathbb{F}_2} = D(b) \cup D(c) \subset \mathrm{Spec} k[a,b,c]/(a^2+bc+\beta)$ and $\sigma_{K,u/\mathbb{F}_2}(\alpha) = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$. Then $\widetilde{\mathrm{Rep}_2(K)}_{u/\mathbb{F}_2} = D(b) \cup D(c) \subset \mathrm{Spec} k[a,b,c,X]/(a^2+bc+\beta, X^2-\beta)$. Identifying K with $k[X]/(X^2-\beta)$, we have $\widetilde{\mathrm{Rep}_2(K)}_{u/\mathbb{F}_2} = D(b) \cup D(c) \subset \mathrm{Spec}\ K[a,b,c]/(a^2+bc+\beta)$. In particular, $\widetilde{\mathrm{Rep}_2(K)}_{u/\mathbb{F}_2} = \mathrm{Rep}_2(K)_{u/\mathbb{F}_2} \otimes_k K$. Remark that $\mathrm{Rep}_2(K)_{u/\mathbb{F}_2} = \mathrm{Rep}_2(K)_{u/\mathbb{F}_2,\alpha}$ and that $\widetilde{\mathrm{Rep}_2(K)}_{u/\mathbb{F}_2} = \widetilde{\mathrm{Rep}_2(K)}_{u/\mathbb{F}_2,\alpha}$.

Let us use the same notation in Remark 6.23. The universal character d'_K on $\mathrm{Rep}'_1(K)$ is characterized by $d'_K(\alpha)$, and it satisfies $d'_K(\alpha)^2 = \beta^2$. Hence we can write $\mathrm{Rep}'_1(K) = \mathrm{Spec} k[x]/(x^2-\beta^2)$ and $d'_K(\alpha) = x$. The universal (a,b)-coefficients with respect to (α, d'_K) on $\mathrm{Ch}_2(K)_{u/\mathbb{F}_2,\alpha}$ satisfies $a(1) = 1, b(1) = 0, a(\alpha) = 0$, and $b(\alpha) = 1$. By the condition $\beta = a(\beta) = a(\alpha^2) = a(\alpha)^2 + b(\alpha)^2 d'_K(\alpha)$, we have $x = \beta$. Thus, we see that $\mathrm{Ch}_2(K)_{u/\mathbb{F}_2,\alpha} = \mathrm{Spec}\ k[x]/(x-\beta) = \mathrm{Spec}\ k$. On the other hand, $\mathrm{Rep}_1(K) = \mathrm{Spec} k[x]/(x^2-\beta) = \mathrm{Spec} K$ because the universal character d_K on $\mathrm{Rep}_1(K)$ satisfies $d_K(\alpha)^2 = \beta$. The $A_1(K)$-module $\Omega_{K/k}$ introduced in Remark 7.32 is isomorphic to the free module $A_1(K)d\alpha = K d\alpha$. Hence $\widetilde{\mathrm{Ch}_2(K)}_{u/\mathbb{F}_2} = \mathrm{Proj}\ S(A_1(K)d\alpha) = \mathrm{Rep}_1(K) = \mathrm{Spec}\ K$. Therefore, the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Rep}_2(K)}_{u/\mathbb{F}_2} & \xrightarrow{p} & \mathrm{Rep}_2(K)_{u/\mathbb{F}_2} \\ \widetilde{\pi}_{K,u/\mathbb{F}_2} \downarrow & & \downarrow \pi_{K,u/\mathbb{F}_2} \\ \widetilde{\mathrm{Ch}_2(K)}_{u/\mathbb{F}_2} & \xrightarrow{q} & \mathrm{Ch}_2(K)_{u/\mathbb{F}_2} \end{array}$$

is identified with

$$\begin{array}{ccc} \mathrm{Rep}_2(K)_{u/\mathbb{F}_2} \otimes_k K & \to & \mathrm{Rep}_2(K)_{u/\mathbb{F}_2} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ K & \to & \mathrm{Spec}\ k, \end{array}$$

which gives a fibre product. The representation $\rho : K \to \mathrm{End}_k(K)$ gives the only equivalence class of 2-dimensional representations of K over k which have unipotent molds over $\mathbb{F}_2$.

Remark 7.34. For understanding the difference between the moduli schemes $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ and $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$, let us pay attention to the morphism $q : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$. Assume that a point $\widetilde{x}$ of $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}$ is mapped to a point x of $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$. We can write $\widetilde{x} = [(\rho, \lambda)]$ and $x = [\rho]$, where $\rho : \Gamma \to \mathrm{M}_2(k(x))$ is a representation with unipotent mold over $\mathbb{F}_2$ on the residue field $k(x)$ of x and $\lambda : \Gamma \to k(\widetilde{x})$ is a character on the residue field $k(\widetilde{x})$ of $\widetilde{x}$ such that (ρ, λ) is a tilde representation with unipotent mold over $\mathbb{F}_2$. It is easy to see that $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}(K) \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}(K)$ is injective for any field K. Hence q is universally injective (or radical) (see [2, Definition 3.5.4]). Then $k(\widetilde{x})$ is a purely inseparable extension of $k(x)$. In that meaning, q is a generalization of purely inseparable extension of fields and it is globally defined.

Although $q : \widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2} \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is surjective, $\widetilde{\mathrm{Ch}_2(\Gamma)}_{u/\mathbb{F}_2}(K) \to \mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}(K)$ is not surjective in general. In the free monoid case, $q : \widetilde{C(m)} \to C(m)$ is described in Example 7.24. When $m = 1$, $q : \mathbb{A}^1_{\mathbb{F}_2} \to \mathbb{A}^1_{\mathbb{F}_2}$ is given by $r_1 \mapsto r_1^2$, where $r_1 = r(\alpha_1)$ and α_1 is the generator of the free monoid $\Upsilon_1 = \langle \alpha_1 \rangle$. Let β be an element of a field k of characteristic 2 such that $\alpha = \sqrt{\beta} \notin k$. Let x be the k-rational point of $C(1) \cong \mathbb{A}^1_{\mathbb{F}_2}$ given by $r_1^2 = r(\alpha_1)^2 = \beta \in k$, and let $\widetilde{x}$ be the $k(\alpha)$-rational point of $\widetilde{C(1)} \cong \mathbb{A}^1_{\mathbb{F}_2}$ given by $r_1 = r(\alpha_1) = \alpha \in k(\alpha)$. Then $\widetilde{x}$ corresponds to x and $k(\alpha)$ is a purely inseparable extension of degree 2 over k (*cf.* Example 7.33). In particular, $\widetilde{C(1)}(k) \to C(1)(k)$ is not surjective, since x is not contained in the image. Remark that if Γ is finitely generated, then the residue field $k(x)$ of a closed point x of $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ is a finite field. In this case, $k(\widetilde{x}) = k(x)$ for the unique point $\widetilde{x}$ lying over x. Note that $q : \widetilde{C(m)} \to C(m)$ induces a bijection of sets $\widetilde{C(m)}(K) \cong C(m)(K)$ if K is an algebraically closed field of characteristic 2 and that q induces a purely inseparable extension of the function fields of degree 2 (see also [13, Remark 3.3]).

Remark 7.35. We have introduced the notion of generalized tilde representations with unipotent mold over $\mathbb{F}_2$ for describing the moduli functors $\widetilde{\mathcal{E}q\mathcal{U}_2}(\Gamma)_{\mathbb{F}_2}$ and $\widetilde{\mathcal{E}q\mathcal{U}_2}(A)_{\mathbb{F}_2}$. However, the moduli functors can also be described as $\widetilde{\mathcal{E}q\mathcal{U}'_2}(\Gamma)_{\mathbb{F}_2}$ and $\widetilde{\mathcal{E}q\mathcal{U}'_2}(A)_{\mathbb{F}_2}$ by using the notion of tilde representations generating sheaves of algebras which define unipotent molds over $\mathbb{F}_2$. More precisely, see §8.

8. Representations in sheaves of algebras

For describing the moduli functor $\mathcal{E}q\mathcal{SS}_2(\Gamma)$ ($\mathcal{E}q\mathcal{U}_2(\Gamma)$, or $\mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$), we introduced the notion of generalized representations with semi-simple mold (unipotent mold, unipotent mold over $\mathbb{F}_2$, respectively) in §4-§6. However, we can also formulate these moduli functors by using representations generating sheaves of $\mathcal{O}_X$-algebras which define molds of rank 2 on a scheme X. In this section, we discuss this formulation for describing the moduli schemes. We also reformulate $\widetilde{\mathcal{E}q\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}$ in §7 by using tilde representations generating sheaves of $\mathcal{O}_X$-algebras which define unipotent molds over $\mathbb{F}_2$. The author needs to say that this section was inspired by the referee.

Definition 8.1. Let Γ be a group or a monoid. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras on a scheme X. We say that a homomorphism $\rho : \Gamma \to \mathcal{A}(X)$ is a *representation in* $\mathcal{A}$ of Γ. For two representations $\rho_1 : \Gamma \to \mathcal{A}_1(X)$ and $\rho_2 : \Gamma \to \mathcal{A}_2(X)$, we say that ρ_1 and ρ_2 are *equivalent* if there exists an isomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ as sheaves of $\mathcal{O}_X$-algebras such that $\phi \circ \rho_1 = \rho_2$. We call a representation $\rho : \Gamma \to \mathcal{A}(X)$ a *representation generating* $\mathcal{A}$ if $\mathcal{O}_X[\rho(\Gamma)] = \mathcal{A}$.

Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras on a scheme X which is locally free of rank 2. We define $\Phi_{\mathcal{A}} : \mathcal{A} \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{A})$ by $a \mapsto (b \mapsto ab)$ for each open subset U of X and for each $a, b \in \mathcal{A}(U)$. Then $\Phi_{\mathcal{A}}$ is injective. Remark that $\mathcal{A}(U)$ is a commutative ring since $\mathcal{A}$ is locally free of rank 2 and $1 \in \mathcal{A}(X)$. For each $x \in X$, choose a neighbourhood U_x of x such that $\mathcal{A}|_{U_x} \cong \mathcal{O}_{U_x}^{\oplus 2}$. By considering the inclusion $\mathcal{A}|_{U_x} \cong \Phi_{\mathcal{A}}|_{U_x}(\mathcal{A}|_{U_x}) \subset \mathrm{End}_{\mathcal{O}_{U_x}}(\mathcal{A}|_{U_x}) \cong \mathrm{M}_2(\mathcal{O}_{U_x})$, we obtain a mold of rank 2 on U_x.

Definition 8.2. If $\Phi_{\mathcal{A}}|_{U_x}(\mathcal{A}|_{U_x})$ is a semi-simple mold (unipotent mold, or unipotent mold over $\mathbb{F}_2$) for each $x \in X$, we say that $\mathcal{A}$ *defines a semi-simple mold (unipotent mold, or unipotent mold over* $\mathbb{F}_2$*, respectively).* This definition does not depend on choices of neighbourhoods U_x of x and isomorphisms $\mathcal{A}|_{U_x} \cong \mathcal{O}_{U_x}^{\oplus 2}$.

For a generalized representation $\{(U_i, \rho_i)\}_{i \in I}$ with semi-simple mold (unipotent mold, unipotent mold over $\mathbb{F}_2$, respectively) of Γ on a scheme X, we define a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras which is a locally free sheaf of rank 2 as follows: Set $\mathcal{A}_i := \mathcal{O}_{U_i}[\rho_i(\Gamma)]$. Let us define an isomorphism $\varphi_{ij} : \mathcal{A}_i|_{U_i \cap U_j} \to \mathcal{A}_j|_{U_i \cap U_j}$ by $\rho_i(\gamma) \mapsto \rho_j(\gamma)$ for each $\gamma \in \Gamma$. It is easy to check that $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij} : \mathcal{A}_i|_{U_i \cap U_j \cap U_k} \to \mathcal{A}_k|_{U_i \cap U_j \cap U_k}$ and that $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ii} = \mathrm{id}$. Hence by using [6, Chap. II, Ex.1.22], we obtain a unique sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras on X (up to isomorphism), together with isomorphisms $\psi_i : \mathcal{A}|_{U_i} \xrightarrow{\sim} \mathcal{A}_i$ such that $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$

for each i, j. Obviously, $\mathcal{A}$ is locally free of rank 2. If $\{(U_i, \rho_i)\}_{i \in I}$ is a generalized representation with semi-simple mold (unipotent mold, or unipotent mold over $\mathbb{F}_2$), then $\mathcal{A}$ defines a semi-simple mold (unipotent mold, or unipotent mold over $\mathbb{F}_2$, respectively). By gluing $\{\rho_i : \Gamma \to \mathrm{M}_2(\mathcal{O}_{U_i}(U_i))\}_{i \in I}$, we have a representation $\rho : \Gamma \to \mathcal{A}(X)$ in $\mathcal{A}$ of Γ such that $\Gamma \stackrel{\rho}{\to} \mathcal{A}(X) \stackrel{\mathrm{res}}{\to} \mathcal{A}(U_i) \stackrel{\psi_i}{\to} \mathcal{A}_i(U_i)$ coincides with ρ_i for each $i \in I$. Then ρ is a representation generating $\mathcal{A}$. Thus each generalized representation $\{(U_i, \rho_i)\}_{i \in I}$ corresponds to a representation $\rho : \Gamma \to \mathcal{A}(X)$ of Γ generating $\mathcal{A}$ on X.

Definition 8.3. Let us define the contravariant functor $\mathcal{E}q\mathcal{SS}'_2(\Gamma)$ from the category of schemes to the category of sets as follows:

$$\begin{array}{cccc} \mathcal{E}q\mathcal{SS}'_2(\Gamma): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ \begin{array}{l} \text{a representation of } \Gamma \text{ generating} \\ \text{a sheaf of } \mathcal{O}_X\text{-algebras } \mathcal{A} \text{ which} \\ \text{is locally free of rank 2 and} \\ \text{defines a semi-simple mold on } X \end{array} \right\} \Big/ \sim . \end{array}$$

We also define the contravariant functors $\mathcal{E}q\mathcal{U}'_2(\Gamma) : (\mathbf{Sch}/\mathbb{Z}[1/2])^{op} \to (\mathbf{Sets})$ and $\mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2} : (\mathbf{Sch}/\mathbb{F}_2)^{op} \to (\mathbf{Sets})$ in the same way.

By the correspondence above, we obtain natural transformations $\sigma_{s.s.} : \mathcal{E}q\mathcal{SS}_2(\Gamma) \to \mathcal{E}q\mathcal{SS}'_2(\Gamma)$, $\sigma_u : \mathcal{E}q\mathcal{U}_2(\Gamma) \to \mathcal{E}q\mathcal{U}'_2(\Gamma)$, and $\sigma_{u/\mathbb{F}_2} : \mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2} \to \mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2}$.

Let us define natural transformations $\tau_{s.s.} : \mathcal{E}q\mathcal{SS}'_2(\Gamma) \to \mathcal{E}q\mathcal{SS}_2(\Gamma)$, $\tau_u : \mathcal{E}q\mathcal{U}'_2(\Gamma) \to \mathcal{E}q\mathcal{U}_2(\Gamma)$, and $\tau_{u/\mathbb{F}_2} : \mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2} \to \mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}$ in the following way. Let ρ be a representation of Γ generating $\mathcal{A}$ on a scheme. Assume that $\mathcal{A}$ defines a semi-simple mold, unipotent mold, or unipotent mold over $\mathbb{F}_2$ on X. For each $x \in X$, choose a neighbourhood U_x such that $\mathcal{A}\mid_{U_x} \cong \mathcal{O}_{U_x}^{\oplus 2}$. Then by considering $\mathcal{A}\mid_{U_x} \cong \Phi_{\mathcal{A}}\mid_{U_x}(\mathcal{A}\mid_{U_x}) \subset \mathrm{End}_{\mathcal{O}_{U_x}}(\mathcal{A}\mid_{U_x}) \cong \mathrm{M}_2(\mathcal{O}_{U_x})$, we have a representation $\rho_x : \Gamma \to \mathrm{M}_2(\mathcal{O}_{U_x})$ with the corresponding mold on U_x. It is easy to check that $\{(U_x, \rho_x)\}_{x \in X}$ is a generalized representation with the corresponding mold on X and that the equivalence class of $\{(U_x, \rho_x)\}_{x \in X}$ is well-defined. The equivalence class of $\{(U_x, \rho_x)\}_{x \in X}$ does not depend on choosing a representative of the equivalence class of $\rho : X \to \mathcal{A}(X)$. This correspondence defines $\tau_{s.s.}$, τ_u, and $\tau_{u/\mathbb{F}_2}$.

It is straightforward to verify that $\tau_{s.s.} \circ \sigma_{s.s.} = 1_{\mathcal{E}q\mathcal{SS}_2(\Gamma)}$ and $\sigma_{s.s.} \circ \tau_{s.s.} = 1_{\mathcal{E}q\mathcal{SS}'_2(\Gamma)}$ and so on. Hence we can obtain the following:

Proposition 8.4. *There are canonical isomorphisms:*

$$\begin{aligned}\mathcal{E}q\mathcal{SS}_2(\Gamma) &\cong \mathcal{E}q\mathcal{SS}'_2(\Gamma),\\ \mathcal{E}q\mathcal{U}_2(\Gamma) &\cong \mathcal{E}q\mathcal{U}'_2(\Gamma),\\ \mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2} &\cong \mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2}.\end{aligned}$$

In particular, $\mathrm{Ch}_2(\Gamma)_{s.s.}$, $\mathrm{Ch}_2(\Gamma)_u$, *and* $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *represent* $\mathcal{E}q\mathcal{SS}'_2(\Gamma)$, $\mathcal{E}q\mathcal{U}'_2(\Gamma)$, *and* $\mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2}$, *respectively.*

In the case of unipotent molds over $\mathbb{F}_2$, we can define another functor $\mathcal{E}q\mathcal{U}''_2(\Gamma)_{\mathbb{F}_2}$:

Definition 8.5. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras which is locally free of rank 2 on a scheme X over $\mathbb{F}_2$. We say that $a \in \mathcal{A}(X)$ is *scalar* if there exists $f \in \mathcal{O}_X(X)$ such that $a = f \cdot 1_{\mathcal{A}}$. We define $\mathcal{E}q\mathcal{U}''_2(\Gamma)_{\mathbb{F}_2}$ by

$$\begin{array}{rcl}\mathcal{E}q\mathcal{U}''_2(\Gamma)_{\mathbb{F}_2}: \quad (\mathbf{Sch}/\mathbb{F}_2)^{op} & \to & (\mathbf{Sets})\\ X & \mapsto & \left\{\begin{array}{l}\text{a representation } \rho \text{ of } \Gamma\\ \text{generating a sheaf of}\\ \mathcal{O}_X\text{-algebras } \mathcal{A} \text{ which is}\\ \text{locally free of rank 2 on } X\\ \text{such that } \rho(\gamma)^2 \text{ is scalar}\\ \text{for each } \gamma \in \Gamma\end{array}\right\}\Big/\sim.\end{array}$$

Proposition 8.6. *There are canonical isomorphisms*

$$\mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2} \cong \mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2} \cong \mathcal{E}q\mathcal{U}''_2(\Gamma)_{\mathbb{F}_2}.$$

Proof. It suffices to prove that $\mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2} \cong \mathcal{E}q\mathcal{U}''_2(\Gamma)_{\mathbb{F}_2}$. For $[\rho : \Gamma \to \mathcal{A}(X)] \in \mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2}(X)$ with a scheme X over $\mathbb{F}_2$, $\rho(\gamma)^2 = \det(\rho(\gamma))\cdot 1_{\mathcal{A}}$ is scalar for each $\gamma \in \Gamma$. Hence $[\rho : \Gamma \to \mathcal{A}(X)] \in \mathcal{E}q\mathcal{U}''_2(\Gamma)_{\mathbb{F}_2}(X)$. Conversely, let $[\rho : \Gamma \to \mathcal{A}(X)] \in \mathcal{E}q\mathcal{U}''_2(\Gamma)_{\mathbb{F}_2}(X)$. For $x \in X$, there exist $\gamma \in \Gamma$ and a neighbourhood U of x such that $\mathcal{A}\,|_U \cong \mathcal{O}_U \cdot 1_U \oplus \mathcal{O}_U \cdot \rho(\gamma)$. Since $\rho(\gamma)^2$ is scalar, $\rho(\gamma)^2 = c \cdot 1_{\mathcal{A}}$ for some $c \in \mathcal{O}_U(U)$. By the Cayley-Hamilton theorem, $\rho(\gamma)^2 - \mathrm{tr}(\rho(\gamma))\rho(\gamma) + \det(\rho(\gamma))I_2 = 0$ on U. Thus we have $\mathrm{tr}(\rho(\gamma)) = 0$ and $\det(\rho(\gamma)) = c$ on U. For any $\gamma' \in \Gamma$, $\rho(\gamma') = aI_2 + b\rho(\gamma)$ on U for some $a, b \in \mathcal{O}_U(U)$. This implies that $\mathrm{tr}(\rho(\gamma')) = a\mathrm{tr}(I_2) + b\mathrm{tr}(\rho(\gamma)) = 0$. Hence $\mathcal{A}$ defines a unipotent mold over $\mathbb{F}_2$ and that $[\rho : \Gamma \to \mathcal{A}(X)] \in \mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2}(X)$. Therefore we have proved that $\mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2} \cong \mathcal{E}q\mathcal{U}''_2(\Gamma)_{\mathbb{F}_2}$. $\square$

Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras which is locally free of rank 2 on a scheme X over $\mathbb{F}_2$. Let $\rho : \Gamma \to \mathcal{A}(X)$ be a representation of Γ generating $\mathcal{A}$, and let $\chi : \Gamma \to \mathcal{O}_X(X)$ be a character. We say that a pair (ρ, χ) is a *tilde representation with unipotent mold over* $\mathbb{F}_2$ *for* Γ *generating* $\mathcal{A}$ *on* X if $\{\rho(\gamma) - \chi(\gamma) \cdot 1_{\mathcal{A}}\}_{\gamma\in\Gamma}$ spans a sub-line bundle of

$\mathcal{A}$ and $(\rho(\gamma)-\chi(\gamma)\cdot 1_{\mathcal{A}})^2=0$ for any $\gamma\in\Gamma$. (Then we can prove that $\mathcal{A}$ defines a unipotent mold over $\mathbb{F}_2$ as in the proof of Proposition 8.8.) For two tilde representations (ρ_1,χ_1) and (ρ_2,χ_2) with unipotent mold over $\mathbb{F}_2$ for Γ on X, we say that they are *equivalent* if $\chi_1=\chi_2$ and there exists an isomorphism $\phi:\mathcal{A}_1\to\mathcal{A}_2$ as sheaves of $\mathcal{O}_X$-algebras such that $\phi\circ\rho_1=\rho_2$, where ρ_i is a homomorphism $\rho_i:\Gamma\to\mathcal{A}_i(X)$ for $i=1,2$.

Definition 8.7.

$$\begin{array}{llll}\widetilde{\mathcal{E}q\mathcal{U}'_2(\Gamma)}_{\mathbb{F}_2}: & (\mathbf{Sch}/\mathbb{F}_2)^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ (\rho,\chi) \;\middle|\; \begin{array}{l} (\rho,\chi)\text{ is a tilde representation} \\ \text{with unipotent mold over }\mathbb{F}_2 \\ \text{for }\Gamma\text{ generating a sheaf of} \\ \mathcal{O}_X\text{-algebras }\mathcal{A}\end{array}\right\} \Big/ \sim .\end{array}$$

Proposition 8.8. *There is a canonical isomorphism*

$$\widetilde{\mathcal{E}q\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}\cong\widetilde{\mathcal{E}q\mathcal{U}'_2(\Gamma)}_{\mathbb{F}_2}.$$

Proof. Let $\{(U_i,\rho_i,\lambda_i)\}_{i\in I}\in\widetilde{\mathcal{E}q\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}(X)$ be a generalized tilde representation with unipotent mold over $\mathbb{F}_2$ for Γ on a scheme X over $\mathbb{F}_2$. Since $\{(U_i,\rho_i)\}_{i\in I}\in\mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}(X)$, we have $[\rho:\Gamma\to\mathcal{A}(X)]\in\mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2}(X)$ by $\mathcal{E}q\mathcal{U}_2(\Gamma)_{\mathbb{F}_2}(X)\cong\mathcal{E}q\mathcal{U}'_2(\Gamma)_{\mathbb{F}_2}(X)$. By gluing $\{(U_i,\lambda_i)\}_{i\in I}$, we can define a character $\chi:\Gamma\to\mathcal{O}_X(X)$ such that $\chi|_{U_i}=\lambda_i$ for $i\in I$. Note that $\det\rho_i(\gamma)=\chi(\gamma)^2$ on U_i. It is easy to see that $(\rho,\chi)\in\widetilde{\mathcal{E}q\mathcal{U}'_2(\Gamma)}_{\mathbb{F}_2}(X)$. This correspondence induces a natural transformation $\widetilde{\sigma_{u/\mathbb{F}_2}}:\widetilde{\mathcal{E}q\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}\to\widetilde{\mathcal{E}q\mathcal{U}'_2(\Gamma)}_{\mathbb{F}_2}$.

Conversely, let $(\rho,\chi)\in\widetilde{\mathcal{E}q\mathcal{U}'_2(\Gamma)}_{\mathbb{F}_2}(X)$. For each point $x\in X$, there exist $\alpha_x\in\Gamma$ and a neighbourhood U_x of x such that $\mathcal{A}|_{U_x}\cong\mathcal{O}_{U_x}\cdot 1_{\mathcal{A}}\oplus\mathcal{O}_{U_x}\cdot\rho(\alpha_x)$. Denote $\Gamma\xrightarrow{\rho}\mathcal{A}|_{U_x}\cong\Phi_{\mathcal{A}}|_{U_x}(\mathcal{A}|_{U_x})\subset\mathrm{End}_{\mathcal{O}_{U_x}}(\mathcal{A}|_{U_x})\cong\mathrm{M}_2(\mathcal{O}_{U_x})$ by ρ_x. By the assumption, $(\rho_x(\alpha_x)-\chi(\alpha_x)I_2)^2=0$. Then $\rho_x(\alpha_x)^2-\chi(\alpha_x)^2I_2=\mathrm{tr}(\rho_x(\alpha_x))\rho_x(\alpha_x)-\det(\rho_x(\alpha_x))I_2-\chi(\alpha_x)^2I_2=0$. Hence $\mathrm{tr}(\rho_x(\alpha_x))=0$ and $\det(\rho_x(\alpha_x))=\chi(\alpha_x)^2$. Since $\{\rho(\gamma)-\chi(\gamma)\cdot 1_{\mathcal{A}}\}_{\gamma\in\Gamma}$ spans a sub-line bundle of $\mathcal{A}$, for each $\gamma\in\Gamma$ there exists $c\in\mathcal{O}_X(U_x)$ such that $\rho(\gamma)-\chi(\gamma)\cdot 1_{\mathcal{A}}=c(\rho(\alpha_x)-\chi(\alpha_x)\cdot 1_{\mathcal{A}})$ on U_x. We have $\rho_x(\gamma)=(\chi(\gamma)-c\chi(\alpha_x))I_2+c\rho_x(\alpha_x)$. Putting $a(\gamma)=\chi(\gamma)-c\chi(\alpha_x)$ and $b(\gamma)=c$, we obtain $\rho_x(\gamma)=a(\gamma)I_2+b(\gamma)\rho_x(\alpha_x)$ and $\chi(\gamma)=a(\gamma)+b(\gamma)\chi(\alpha_x)$. Thereby $\mathrm{tr}(\rho_x(\gamma))=0$ and $\det\rho_x(\gamma)=\chi(\gamma)^2$ for each $\gamma\in\Gamma$. It is easy to check that $\mathcal{A}$ defines a unipotent mold over $\mathbb{F}_2$ and that $\{(U_x,\rho_x,\chi|_{U_x})\}_{x\in X}\in\widetilde{\mathcal{E}q\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}(X)$. Therefore this correspondence induces a natural transformation $\widetilde{\tau_{u/\mathbb{F}_2}} : \widetilde{\mathcal{E}q\mathcal{U}'_2(\Gamma)}_{\mathbb{F}_2} \to \widetilde{\mathcal{E}q\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}$.

It is easy to see that $\widetilde{\tau_{u/\mathbb{F}_2}} \circ \widetilde{\sigma_{u/\mathbb{F}_2}} = 1_{\widetilde{\mathcal{E}q\mathcal{U}_2(\Gamma)}_{\mathbb{F}_2}}$ and that $\widetilde{\sigma_{u/\mathbb{F}_2}} \circ \widetilde{\tau_{u/\mathbb{F}_2}} = 1_{\widetilde{\mathcal{E}q\mathcal{U}'_2(\Gamma)}_{\mathbb{F}_2}}$. This completes the proof. $\square$

In the case of representations of an associative algebra over a commutative ring, we have similar results as the group or monoid cases.

Definition 8.9. Let A be an associative algebra over a commutative ring R. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras on an R-scheme X. We say that an R-homomorphism $\rho : A \to \mathcal{A}(X)$ is a *representation in* $\mathcal{A}$ of A. For two representations $\rho_1 : A \to \mathcal{A}_1(X)$ and $\rho_2 : A \to \mathcal{A}_2(X)$, we say that ρ_1 and ρ_2 are *equivalent* if there exists an isomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ as sheaves of $\mathcal{O}_X$-algebras such that $\phi \circ \rho_1 = \rho_2$. We call a representation $\rho : A \to \mathcal{A}(X)$ a *representation generating* $\mathcal{A}$ if $\mathcal{O}_X[\rho(A)] = \mathcal{A}$.

In the same way as group or monoid cases, we define $\mathcal{E}q\mathcal{S}\mathcal{S}'_2(A)$, $\mathcal{E}q\mathcal{U}'_2(A)$, and $\mathcal{E}q\mathcal{U}'_2(A)_{\mathbb{F}_2}$. Similarly, we have

Proposition 8.10. *There are canonical isomorphisms:*

$$\begin{aligned} \mathcal{E}q\mathcal{S}\mathcal{S}_2(A) &\cong \mathcal{E}q\mathcal{S}\mathcal{S}'_2(A), \\ \mathcal{E}q\mathcal{U}_2(A) &\cong \mathcal{E}q\mathcal{U}'_2(A), \\ \mathcal{E}q\mathcal{U}_2(A)_{\mathbb{F}_2} &\cong \mathcal{E}q\mathcal{U}'_2(A)_{\mathbb{F}_2}. \end{aligned}$$

Hence we can conclude that $\mathrm{Ch}_2(A)_{s.s.}$, $\mathrm{Ch}_2(A)_u$, and $\mathrm{Ch}_2(A)_{u/\mathbb{F}_2}$ represent $\mathcal{E}q\mathcal{S}\mathcal{S}'_2(A)$, $\mathcal{E}q\mathcal{U}'_2(A)$, and $\mathcal{E}q\mathcal{U}'_2(A)_{\mathbb{F}_2}$, respectively.

Definition 8.11. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras which is locally free of rank 2 on an R-scheme X. Let $\rho : A \to \mathcal{A}(X)$ be a representation generating $\mathcal{A}$ on X, and let $\chi : A \to \mathcal{O}_X(X)$ be an R-homomorphism. We say that (ρ, χ) is a *tilde representation with unipotent mold over* $\mathbb{F}_2$ *for* A *generating* $\mathcal{A}$ on X if $\{\rho(c) - \chi(c) \cdot 1_{\mathcal{A}}\}_{c \in A}$ spans a sub-line bundle of $\mathcal{A}$ and $(\rho(c) - \chi(c) \cdot 1_{\mathcal{A}})^2 = 0$ for any $c \in A$.

We can also define $\widetilde{\mathcal{E}q\mathcal{U}'_2(A)}_{\mathbb{F}_2}$. Similarly, we have

Proposition 8.12. *There are a canonical isomorphism:*

$$\widetilde{\mathcal{E}q\mathcal{U}_2(A)}_{\mathbb{F}_2} \cong \widetilde{\mathcal{E}q\mathcal{U}'_2(A)}_{\mathbb{F}_2}.$$

In particular, $\widetilde{\mathrm{Ch}_2(A)}_{u/\mathbb{F}_2}$ *represents* $\widetilde{\mathcal{E}q\mathcal{U}'_2(A)}_{\mathbb{F}_2}$.

9. Appendix: Discriminants

In this section we deal with the discriminant locus of the representation variety of degree 2. The discriminant locus is exactly the subset consisting of representations which are not absolutely irreducible. We describe the absolutely irreducible representation part of the representation variety of degree 2 explicitly (*cf.* [15] or [16]).

Definition 9.1 ([15], [16])**.** Let R be a commutative ring. For $A, B \in \mathrm{M}_2(R)$ we define the *discriminant* $\Delta(A, B)$ by

$$\begin{aligned}\Delta(A,B) \;\; := \;\; & \mathrm{tr}(A)^2\det(B) + \mathrm{tr}(B)^2\det(A) + \mathrm{tr}(AB)^2 \\ & -\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) - 4\det(A)\det(B).\end{aligned}$$

From the definition we see that $\Delta(A, B) = \Delta(B, A)$. If $A, B \in \mathrm{GL}_2(R)$, then $\Delta(A, B) = \det(A)\det(B)\mathrm{tr}(ABA^{-1}B^{-1} - I_2)$.

Remark 9.2. The discriminant $\Delta(A, B)$ above is closely related to the discriminant in [10]. For $A_1, A_2, A_3, A_4 \in \mathrm{M}_2(R)$ we define the *discriminant* of degree 2 in [10] by

$$\Delta(A_1, A_2, A_3, A_4) := \det\begin{pmatrix} a(1)_{11} & a(1)_{12} & a(1)_{21} & a(1)_{22} \\ a(2)_{11} & a(2)_{12} & a(2)_{21} & a(2)_{22} \\ a(3)_{11} & a(3)_{12} & a(3)_{21} & a(3)_{22} \\ a(4)_{11} & a(4)_{12} & a(4)_{21} & a(4)_{22} \end{pmatrix},$$

where $A_i = \begin{pmatrix} a(i)_{11} & a(i)_{12} \\ a(i)_{21} & a(i)_{22} \end{pmatrix}$ for $i = 1, 2, 3, 4$. Then we have

$$\Delta(A, B) = -\Delta(I_2, A, B, AB).$$

Note that $\Delta(A_1, A_2, A_3, A_4) \in R^\times$ if and only if $\{A_1, A_2, A_3, A_4\}$ is an R-basis of $\mathrm{M}_2(R)$.

Lemma 9.3. *Let k be a field. Assume that $\mathcal{A} \subseteq \mathrm{M}_2(k)$ is a subalgebra over k. Then $\mathcal{A} \neq \mathrm{M}_2(k)$ if and only if $\mathcal{A}$ is commutative or there exists a 1-dimensional $\mathcal{A}$-invariant subspace of k^2.*

Proof. The "if" part is easy. We only need to prove the "only if" part. Suppose that $\mathcal{A} \neq \mathrm{M}_2(k)$. If $\mathcal{A}$ has no nontrivial invariant subspace, then k^2 is a simple $\mathcal{A}$-module. Since the Jacobson radical $\mathrm{Jac}\mathcal{A}$ is equal to $\cap_{M:\,\mathrm{simple}}\mathrm{Ann}M = 0$, the algebra $\mathcal{A}$ is semi-simple. From the Wedderburn Theorem we see that $\mathcal{A}$ is a product of the full matrix rings over division algebras over k. Because $\mathcal{A}$ has a faithful 2-dimensional simple module and $\dim\mathcal{A} \leq 3$, the algebra $\mathcal{A}$ is isomorphic to a quadratic extension of k. Hence $\mathcal{A}$ is commutative. $\square$

Proposition 9.4. *Let k be a field. Suppose that $A, B \in \mathrm{M}_2(k)$ and $\mathcal{A}$ is the k-subalgebra of $\mathrm{M}_2(k)$ generated by A and B. Then the following conditions are equivalent:*

(i) $\Delta(A, B) = 0$.
(ii) $\mathcal{A} \neq \mathrm{M}_2(k)$.
(iii) *A and B are commutative or A and B have a common invariant subspace of dimension* 1.

Proof. The part (ii) $\Leftrightarrow$ (iii) follows from Lemma 9.3. The part (ii) $\Rightarrow$ (i) follows from that $\{I_2, A, B, AB\}$ is not a basis of $\mathrm{M}_2(k)$ and that $\Delta(A, B) = -\Delta(I_2, A, B, AB) = 0$ by Remark 9.2. We only have to prove (i) $\Rightarrow$ (ii).

We assume that $\Delta(A, B) = 0$. First, we show that AB and BA can be expressed as linear combinations of $\{I_2, A, B\}$. Here let us consider the case AB. From $\Delta(A, B) = 0$, we see that $\{I_2, A, B, AB\}$ is linearly dependent. Hence there exist $c_i \in k$ $(1 \leq i \leq 4)$ such that $c_1 I_2 + c_2 A + c_3 B + c_4 AB = 0$ and some $c_i \neq 0$. If $c_4 \neq 0$, then the claim is true. If $c_4 = 0$, then either $c_2 \neq 0$ or $c_3 \neq 0$ holds. When $c_2 \neq 0$, the matrix A is expressed as a linear combination of I_2 and B, and hence AB can be expressed as a polynomial of B. By the Cayley-Hamilton Theorem, AB can be expressed as a linear combination of I_2 and B. We can also prove the claim for the $c_3 \neq 0$ case. Thus we have shown that AB can be expressed as a linear combination of $\{I_2, A, B\}$. We can also prove the BA case in the same way.

Next, we show that any monomial of A and B can be expressed as a linear combination of $\{I_2, A, B\}$. This implies that $\mathcal{A} \neq \mathrm{M}_2(k)$. We prove the claim by induction on the length of monomials. The length 0, 1 and 2 cases are true. Suppose that the length $n-1$ case is true for $n \geq 3$. Let X be a monomial whose length is n. If X has a subsequence AB or BA, then X can be reduced to the the length $n-1$ case from the above claim. If X has a subsequence AA or BB, then from the Cayley-Hamilton Theorem we also see that X can be reduced to the length $n-1$ case. This completes the proof. $\square$

Corollary 9.5. *Let k be a field. Suppose that $\mathcal{A}$ is a k-subalgebra of $\mathrm{M}_2(k)$. Then the following conditions are equivalent:*

(i) $\Delta(A, B) = 0$ *for each* $A, B \in \mathcal{A}$.
(ii) $\mathcal{A} \neq \mathrm{M}_2(k)$, *or equivalently the $\mathcal{A}$-module k^2 is not absolutely irreducible.*
(iii) *$\mathcal{A}$ is commutative or $\mathcal{A}$ has an invariant subspace of dimension* 1.

Proof. The part (ii) $\Leftrightarrow$ (iii) follows from Lemma 9.3. Suppose that (ii) holds. For any $A, B \in \mathcal{A}$, the matrices A and B do not generate the full matrix ring $\mathrm{M}_2(k)$, so $\Delta(A, B) = 0$ from the above proposition. This shows that (ii) $\Rightarrow$ (i) holds.

We now prove (i) $\Rightarrow$ (ii). Assume that (i) holds. Suppose that $\mathcal{A} = \mathrm{M}_2(k)$. For

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathcal{A},$$

we have $\Delta(A, B) = 1 \neq 0$. This is a contradiction. Therefore we have proved (i) $\Rightarrow$ (ii). $\square$

Here we introduce another invariant.

Definition 9.6. Let R be a commutative ring. For $A, B, C \in \mathrm{M}_2(R)$ we define $\tau(A, B, C)$ by

$$\tau(A, B, C) := \mathrm{tr}(ABC) - \mathrm{tr}(ACB)$$

or equivalently,

$$\tau(A, B, C) = 2\mathrm{tr}(ABC) - \mathrm{tr}(A)\mathrm{tr}(BC) - \mathrm{tr}(B)\mathrm{tr}(CA) - \mathrm{tr}(C)\mathrm{tr}(AB) \\ +\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C).$$

Remark 9.7. The above τ is closely related to the discriminant defined in [10]. Indeed, $\tau(A, B, C) = \Delta(A, B, C, I_2)$ holds for $A, B, C \in \mathrm{M}_2(R)$.

Definition 9.8. Let k be a field. Pick $x \in k^2 - \{0\}$. We denote by $\overline{x}$ the equivalence class containing x in $\mathbb{P}^1_k := (k^2 - \{0\})/k^\times$. For $A \in \mathrm{M}_2(k)$ we say that $\overline{x}$ is an A-*fixed point* if x is an eigenvector of A. In particular if $A \in \mathrm{GL}_2(R)$, then A can be regarded as an automorphism of $\mathbb{P}^1_k$, and so $\overline{x}$ is an A-fixed point if and only if $\overline{x}$ is fixed by A as a point of $\mathbb{P}^1_k$.

Remark 9.9. If $A \in \mathrm{M}_2(k)$ is not a scalar matrix, then A has at most two fixed points in $\mathbb{P}^1_k$.

Proposition 9.10. *Let k be a field. Suppose that $A, B, C \in \mathrm{M}_2(k)$ and that $\mathcal{A}$ is a k-subalgebra of $\mathrm{M}_2(k)$ generated by A, B, C. Then the following conditions are equivalent.*

(i) $\Delta(A, B) = \Delta(B, C) = \Delta(C, A) = \tau(A, B, C) = 0$.

(ii) $\mathcal{A} \neq \mathrm{M}_2(k)$*, or equivalently the $\mathcal{A}$-module k^2 is not absolutely irreducible.*

Furthermore, if $A, B, C \in \mathrm{GL}_2(k)$, then the following condition is also equivalent to the above two conditions.

(iii) $\Delta(A, B) = \Delta(B, C) = \Delta(C, A) = \Delta(AB, C) = 0$.

Proof. Since we may replace k with an algebraic closure $\overline{k}$ of k, we assume that k is an algebraic closed field from the beginning. Note that $\mathcal{A} \neq \mathrm{M}_2(k)$ if and only if the $\mathcal{A}$-module k^2 is not irreducible when $k = \overline{k}$.

(ii) $\Rightarrow$ (i) The $\mathcal{A}$-module k^2 is not irreducible, so there exists $P \in \mathrm{GL}_2(k)$ such that

$$P^{-1}AP = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, P^{-1}BP = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, P^{-1}CP = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

This immediately implies that $\mathrm{tr}(ABC) = \mathrm{tr}(ACB)$. Hence we have $\tau(A, B, C) = 0$. By Corollary 9.5, $\Delta(A, B) = \Delta(B, C) = \Delta(C, A) = 0$.

(i) $\Rightarrow$ (ii) If one of A, B, C is a scalar matrix, then (i) $\Rightarrow$ (ii) follows from Proposition 9.4. Hence we may assume that none of A, B, C are scalar matrices. Note that if $X \in \mathrm{M}_2(k)$ is not a scalar matrix, then X has at most two fixed points in $\mathbb{P}^1_k$. Suppose that one of A, B, C has exactly one fixed point in $\mathbb{P}^1_k$. Then the others have the same fixed point in $\mathbb{P}^1_k$ since it follows from $\Delta(A, B) = \Delta(B, C) = \Delta(C, A) = 0$ and Proposition 9.4 that k^2 is not an irreducible module over the subalgebras generated by any two of A, B, C. Hence the $\mathcal{A}$-module k^2 is not irreducible.

Now let us consider the case that each of A, B, C has exactly two fixed points in $\mathbb{P}^1_k$. If A, B, C have a common fixed point, then we see that (i) $\Rightarrow$ (ii). Suppose that A, B, C have no common fixed point. By $\Delta(A, B) = \Delta(B, C) = \Delta(C, A) = 0$, we may assume that A has eigenvectors u and v, B has v and w, and C has w and u. With respect to the basis $\{u, v\}$, the matrices A, B, C have the following forms:

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & 0 \\ b_3 & b_4 \end{pmatrix}, C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_4 \end{pmatrix}.$$

Since A, B, C have no common eigenvector, $b_3 \neq 0$ and $c_2 \neq 0$. Hence we have $\mathrm{tr}(ABC) = a_1b_1c_1 + a_4b_3c_2 + a_4b_4c_4$ and $\mathrm{tr}(ACB) = a_1b_1c_1 + a_1b_3c_2 + a_4b_4c_4$. Therefore $\tau(A, B, C) = (a_4 - a_1)b_3c_2 \neq 0$, since A is not a scalar matrix. This is a contradiction. Therefore we have shown that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) This follows from Corollary 9.5.

(iii) $\Rightarrow$ (ii) In the same way as the discussion above in the (i) $\Rightarrow$ (ii) part, we only need to consider the case that A, B, C have exactly two fixed points in $\mathbb{P}^1_k$. Suppose that A, B, C have no fixed point. From the assumption that $\Delta(A, B) = \Delta(B, C) = \Delta(C, A) = 0$ we may assume that A has eigenvectors u and v, B has v and w, and C has w and u, where u, v, and w are distinct up to scalar multiplication. The assumption that $\Delta(AB, C) = 0$ implies that AB and C have a common eigenvector. It is w or u. If w is an eigenvector of AB, then it is also an eigenvector of A because $B \in \mathrm{GL}_2(k)$. This is a contradiction. If u is an eigenvector of AB, then it is also an eigenvector of B because $A \in \mathrm{GL}_2(k)$. This is also a contradiction. Hence the matrices A, B, C have a common eigenvector, which implies that $\mathcal{A} \neq \mathrm{M}_2(k)$. $\square$

From the above proposition we obtain the following proposition.

Proposition 9.11. *Let Γ be a group with a subset $G = \{\alpha_i\}_{i\in I}$ generating Γ. Assume that the index set I is a totally ordered set. The air part $\mathrm{Rep}_2(\Gamma)_{\mathrm{air}}$ of the representation variety of degree 2 for Γ is equal to*

$$\bigcup_{i<j} D(\Delta(\sigma_\Gamma(\alpha_i), \sigma_\Gamma(\alpha_j))) \cup \bigcup_{i<j<k} D(\tau(\sigma_\Gamma(\alpha_i), \sigma_\Gamma(\alpha_j), \sigma_\Gamma(\alpha_k)))$$

or

$$\bigcup_{i<j} D(\Delta(\sigma_\Gamma(\alpha_i), \sigma_\Gamma(\alpha_j))) \cup \bigcup_{i<j<k} D(\Delta(\sigma_\Gamma(\alpha_i\alpha_j), \sigma_\Gamma(\alpha_k))).$$

Here σ_Γ is the universal representation and $D(*)$ is the open subset where $*$ does not vanish.

Proof. Let $k(x)$ be the residue field of a point x of $\mathrm{Rep}_2(\Gamma)$. Note that $x \in \mathrm{Rep}_2(\Gamma)_{\mathrm{air}}$ if and only if $k(x)[\sigma_\Gamma(\Gamma)] = \mathrm{M}_2(k(x))$. By the Cayley-Hamilton theorem, we have $\rho(\alpha^{-1}) \in k(x)[\rho(\alpha)]$ for $\alpha \in \Gamma$. Since $\dim_{k(x)} \mathrm{M}_2(k(x)) = 4$, we see that $x \in \mathrm{Rep}_2(\Gamma)_{\mathrm{air}}$ if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in G$ which generate $\mathrm{M}_2(k(x))$ as a $k(x)$-algebra. Hence we can verify the statement by Proposition 9.10. $\square$

In a similar way, we obtain the following proposition.

Proposition 9.12. *Let Γ be a monoid with a subset $G = \{\alpha_i\}_{i\in I}$ generating Γ. Assume that the index set I is a totally ordered set. The air part $\mathrm{Rep}_2(\Gamma)_{\mathrm{air}}$ of the representation variety of degree 2 for Γ is equal to*

$$\bigcup_{i<j} D(\Delta(\sigma_\Gamma(\alpha_i), \sigma_\Gamma(\alpha_j))) \cup \bigcup_{i<j<k} D(\tau(\sigma_\Gamma(\alpha_i), \sigma_\Gamma(\alpha_j), \sigma_\Gamma(\alpha_k))).$$

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