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VIRTUAL HODGE POLYNOMIALS OF THE MODULI SPACES OF REPRESENTATIONS OF DEGREE 2 FOR FREE MONOIDS

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ABSTRACT. In this paper we study the topology of the moduli spaces of representations of degree 2 for free monoids. We calculate the virtual Hodge polynomials of the character varieties for several types of 2-dimensional representations. Furthermore, we count the number of isomorphism classes for each type of 2-dimensional representations over any finite field $\mathbb{F}_q$, and show that the number coincides with the virtual Hodge polynomial evaluated at q.

1. INTRODUCTION

The moduli spaces of representations have been studied in various contexts (see, for example, [7, 11, 24, 25]). In order to study the representation theory over schemes, the first author defined the representation variety and the character variety over $\mathbb{Z}$ of absolutely irreducible representations for groups or monoids in [18]. Although they are of great importance in the representation theory over schemes, they are difficult to analyze. To overcome this difficulty, the first author introduced representations with Borel mold and studied the representation variety and the character variety of representations with Borel mold in [19]. Furthermore, we studied the topology of these moduli spaces of representations with Borel mold for free monoids over the field $\mathbb{C}$ of complex numbers in [21].

Let us introduce a general framework for the representation theory over schemes for free monoids (cf. [20, 21]). Let $\mathrm{Rep}_n(m) = (\mathrm{M}_n)^m$ the product of m-copies of the full $n \times n$ matrix ring, which is the representation variety of degree n over $\mathbb{Z}$ for the free monoid with m generators. There is a decomposition of $\mathrm{Rep}_n(m)$ by locally closed subschemes

$$\mathrm{Rep}_n(m) = \bigcup_{h=1}^{n^2} \mathrm{Rep}_n(m)_{\mathrm{rk}h},$$

where $\mathrm{Rep}_n(m)_{\mathrm{rk}h}$ is the moduli space of representations which generate a subalgebra of rank h in the matrix algebra M_n. The open subscheme $\mathrm{Rep}_n(m)_{\mathrm{rk}n^2}$ is the moduli space $\mathrm{Rep}_n(m)_{air}$ of absolutely irreducible representations

$$\mathrm{Rep}_n(m)_{air} = \mathrm{Rep}_n(m)_{\mathrm{rk}n^2}.$$

In degree 2 case we can study these moduli spaces in more detail as in [20]. In particular, $\mathrm{Rep}_2(m)_{\mathrm{rk}3}$ is the moduli space $\mathrm{Rep}_2(m)_B$ of representations with Borel mold. We have two subschemes of $\mathrm{Rep}_2(m)_{\mathrm{rk}2}$. One is the moduli space $\mathrm{Rep}_2(m)_{ss}$ of semi-simple representations, and the other is the moduli space $\mathrm{Rep}_2(m)_u$ of unipotent representations. Then $\mathrm{Rep}_2(m)_u$ is

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closed in $\mathrm{Rep}_2(m)_{\mathrm{rk2}}$, and $\mathrm{Rep}_2(m)_{ss}$ is its open complement. We can regard $\mathrm{Rep}_2(m)_{\mathrm{rk1}}$ as the moduli space $\mathrm{Rep}_2(m)_{sc}$ of scalar representations. The conjugate action of matrices induces an action of the group scheme PGL_2 on $\mathrm{Rep}_2(m)_*$ for $* = air, B, ss, u, sc$. The character variety $\mathrm{Ch}_2(m)_*$ is defined as

$$\mathrm{Ch}_2(m)_* = \mathrm{Rep}_2(m)_*/\mathrm{PGL}_2.$$

In [18], [19], and [20], the first author showed that $\mathrm{Ch}_2(m)_*$ is a universal geometric quotient of $\mathrm{Rep}_2(m)_*$ by PGL_2 for $* = air, B, ss, u, sc$. (For $* = u$, we need to divide $\mathrm{Rep}_2(m)_u$ into two parts: the $\mathbb{Z}[1/2]$-part and the $\mathbb{F}_2$-part. More precisely, see [20].)

In this paper we study the topology of the moduli spaces of representations of degree 2 for free monoids. We calculate the integral and rational cohomology groups of representation varieties $\mathrm{Rep}_2(m)_*$ and character varieties $\mathrm{Ch}_2(m)_*$ over the field $\mathbb{C}$ of complex numbers for $* = ss, u, sc$. Also, we give the virtual Hodge polynomial of $\mathrm{Rep}_2(m)_*$ and $\mathrm{Ch}_2(m)_*$ for $* = ss, u, sc, air$. See §5 for the definition of virtual Hodge polynomials. In this paper we obtain the following theorems:

Theorem 1.1 (Theorem 5.8)**.** *The virtual Hodge polynomial of* $\mathrm{Rep}_2(m)_{air}(\mathbb{C})$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathrm{Rep}_2(m)_{air}(\mathbb{C}))(z) &= (1-z^m)(1-z^{m-1}),\\ \mathrm{VHP}_c(\mathrm{Rep}_2(m)_{air}(\mathbb{C}))(z) &= z^{2m+1}(z^m-1)(z^{m-1}-1).\end{aligned}$$

Theorem 1.2 (Theorem 5.18)**.** *The virtual Hodge polynomial of* $\mathrm{Ch}_2(m)_{air}(\mathbb{C})$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathrm{Ch}_2(m)_{air}(\mathbb{C}))(z) &= \tfrac{(1-z^m)(1-z^{m-1})}{1-z^2},\\ \mathrm{VHP}_c(\mathrm{Ch}_2(m)_{air}(\mathbb{C}))(z) &= \tfrac{z^{2m}(z^m-1)(z^{m-1}-1)}{z^2-1}.\end{aligned}$$

These results can be obtained by considering the mixed Hodge structures on the rational cohomology of $\mathrm{Rep}_2(m)_*$ for $* = B, ss, u, sc$ and calculating these virtual Hodge polynomials.

Furthermore, we count the numbers of $\mathbb{F}_q$-valued points of the moduli spaces of representations of degree 2 for free monoids. In particular, the number of $\mathbb{F}_q$-valued points of $\mathrm{Rep}_2(m)_{air}$ is the number of absolutely irreducible representations of degree 2 over $\mathbb{F}_q$ for the free monoid with m generators, and the number of $\mathbb{F}_q$-valued points of $\mathrm{Ch}_2(m)_{air}$ is the number of isomorphism classes of such representations.

Theorem 1.3 (cf. Theorems 6.10 and 6.11)**.** *For any finite field* $\mathbb{F}_q$*, the number of absolutely irreducible representations of degree* 2 *over* $\mathbb{F}_q$ *for the free monoid with* m *generators is given by*

$$|\mathrm{Rep}_2(m)_{air}(\mathbb{F}_q)| = \mathrm{VHP}_c(\mathrm{Rep}_2(m)_{air}(\mathbb{C}))(q),$$

and the number of isomorphism classes of such representations is given by

$$|\mathrm{Ch}_2(m)_{air}(\mathbb{F}_q)| = \mathrm{VHP}_c(\mathrm{Ch}_2(m)_{air}(\mathbb{C}))(q).$$

Note that if there exists a polynomial with integer coefficients which counts the number of $\mathbb{F}_q$-valued points of a separated scheme X of finite type over $\mathbb{Z}$, then $|X(\mathbb{F}_q)| = \mathrm{VHP}_c(X(\mathbb{C}))(q)$. For more details, see [9, §6]. We should mention that these results coincide with the calculation of [23, Example 7.2]. Reineke has already calculated the virtual Hodge polynomials of $\mathrm{Ch}_2(m)_{air}(\mathbb{C})$.

By these results, we have the following main theorem of this paper:

Theorem 1.4 (Theorem 6.14)**.** *The number of* $\mathbb{F}_q$*-valued points of* $\mathrm{Ch}_2(m)$ *is given by*

$$\begin{aligned}|\mathrm{Ch}_2(m)(\mathbb{F}_q)| &= |\mathrm{Ch}_2(m)_{\mathrm{air}}(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_{ss}(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_{sc}(\mathbb{F}_q)|\\ &= \frac{q^{2m+2}(q^{2m-3}-q^{m-2}-q^{m-3}+1)}{q^2-1}.\end{aligned}$$

In particular, the virtual Hodge polynomial of $\mathrm{Ch}_2(m)$ *is given by*

$$\mathrm{VHP}_c(\mathrm{Ch}_2(m))(z) = \frac{z^{2m+2}(z^{2m-3} - z^{m-2} - z^{m-3} + 1)}{z^2 - 1}.$$

In the present paper, we deal with only 2-dimensional representations of free monoids. However, our strategy is available for another groups or monoids. We can calculate the numbers of equivalence classes of absolutely irreducible 2-dimensional representations over $\mathbb{F}_q$ for finitely generated groups or monoids Γ by calculating the number of the $\mathbb{F}_q$-valued points of the representation variety $\mathrm{Rep}_2(\Gamma)$ and the others $\mathrm{Rep}_2(\Gamma)_*$ for $* = B, s.s., u, sc$. If there are polynomials with integer coefficients which count the numbers of the $\mathbb{F}_q$-valued points of the reprensetation variety $\mathrm{Rep}_2(\Gamma)$ and the others $\mathrm{Rep}_2(\Gamma)_*$, then the polynomials coincide with virtual Hodge polynomials of the corresponding schemes.

For 3-dimensional representations, we can also calculate the virtual Hodge polynomials of the character varieties of absolutely irreducible representations for finitely generated free monoids. Indeed, there are 26 types of subalgebras of the full matrix ring of degree 3, and we can calculate the virtual Hodge polynomial of the representation variety $\mathrm{Rep}_3(m)_*$ associated to each subalgebra of M_3 in the same way as 2-dimensional representations.

Here we should point out the results of [3], [13], [15], [16], [23], and so on. After we wrote the present paper, we found out their papers. Their results are very much related to our paper. Our strategy to calculate the numbers of $\mathbb{F}_q$-valued points and the virtual Hodge polynomials of the representation varieties and the character varieties is essentially same as [3] and [13]. Moreover, it is much easier to calculate the virtual Hodge polynomials of $\mathrm{Ch}_n(m)_{air} := \mathrm{Rep}_n(m)_{air}/\mathrm{PGL}_n$ by the method of [23] than to calculate them by our strategy. However, we believe that it is worth publishing our results. Not only are our objects $\mathrm{Rep}_2(m)_*$ and $\mathrm{Ch}_2(m)_*$ different from the $\mathrm{SL}_2(\mathbb{C})$-character varieties of free groups, but also we give a geometric meaning to their stratifications of the representation varieties in [3]. Each stratification represents a certain moduli functor which is described in terms of representation theory. This interpretation allows us to overview the representation varieties and the character varieties from viewpoints of algebraic geometry, algebraic topology, representation theory, and so on. Furthermore, as far as we know, our strategy is the only way to calculate the virtual Hodge polynomial of $\mathrm{Ch}_2(m)$.

The organization of this paper is as follows: In §2 we study the representation variety and the character variety of semi-simple representations. We give descriptions of these moduli spaces. Then we calculate the integral and rational cohomology groups of them. In §3 we define the representation variety and the character variety of unipotent representations of degree 2. Then we give descriptions and calculate the cohomology groups of them. In §4 we define and study the moduli spaces of scalar representations. In §5 we study the virtual Hodge polynomials of the representation varieties and the character varieties of representations of degree 2 for free monoid. Then we prove Theorem 1.1 and Theorem 1.2. In §6 we count the number of $\mathbb{F}_q$-valued points of the moduli spaces, and prove Theorem 1.3 and Theorem 1.4.

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ be the ring of integers, the field of rational numbers, the field of real numbers, the field of complex numbers, respectively. In this paper we denote by $H^*(X)$ the integral cohomology groups, and by $H^*(X;\mathbb{Q})$ the rational cohomology groups of a space X. For a graded module V over $\mathbb{Z}$ or $\mathbb{Q}$, we denote by $\Lambda(V)$ the free commutative graded algebra on V.

2. The moduli spaces of semi-simple representations

In this section we study the moduli spaces related to semi-simple representations. We give descriptions for the moduli spaces and calculate the cohomology groups of them.

2.1. **Descriptions for the moduli spaces of semi-simple representations.** Let K be an algebraically closed field. For $(A_1,\dots,A_m)\in(\mathrm{M}_n(K))^m$, we write $A_k=(a_{ij}(k))_{ij}$. Then we define $a_{ij}\in K^m$ by $a_{ij}=(a_{ij}(1),\dots,a_{ij}(m))$. We denote by D_n the K-subalgebra of $\mathrm{M}_n(K)$ consisting of diagonal matrices. Let $\mathrm{SS}_n(m)$ be the subspace of $(\mathrm{D}_n)^m$ given by

$$\mathrm{SS}_n(m)=\{(A_1,\dots,A_m)\in(\mathrm{D}_n)^m|\ A_1,\dots,A_m \text{ generate } \mathrm{D}_n \text{ as a } K\text{-algebra}\}.$$

We define the moduli space $\mathrm{Rep}_n(m)_{ss}(K)$ of semi-simple representations by

$$\begin{aligned}&\mathrm{Rep}_n(m)_{ss}(K)\\ =\ &\{(A_1,\dots,A_m)\in(\mathrm{M}_n(K))^m|\ (PA_1P^{-1},\dots,PA_mP^{-1})\in\mathrm{SS}_n(m) \text{ for some } P\in\mathrm{PGL}_n(K)\}.\end{aligned}$$

Note that $\mathrm{SS}_n(m)$ is a subspace of $\mathrm{Rep}_n(m)_{ss}(K)$. Let $F_n(K^m)$ be the configuration space of distinct ordered n-points in K^m:

$$F_n(K^m)=\{(p_1,\dots,p_n)\in(K^m)^n|\ p_i\neq p_j \text{ for } i\neq j\}.$$

For $p=(p_1,\dots,p_n)\in(K^m)^n$, we set

$$\varphi(p)=(A_1,\dots,A_m)\in(\mathrm{D}_n)^m$$

where $a_{ij}=p_i$ if $i=j$, and $a_{ij}=0$ if $i\neq j$. Then we obtain an isomorphism $\varphi:(K^m)^n\xrightarrow{\cong}(\mathrm{D}_n)^m$. By [21, Lemma 3.3], we obtain the following lemma.

Lemma 2.1. *The map φ induces an isomorphism $F_n(K^m)\xrightarrow{\cong}\mathrm{SS}_n(m)$ of smooth algebraic varieties. So we can regard $F_n(K^m)$ as a subspace of $\mathrm{Rep}_n(m)_{ss}(K)$.*

Let Σ_n be the symmetric group on n-letters. We regard Σ_n as a subgroup of $\mathrm{PGL}_n(K)$ consisting of permutation matrices. Let T_n be the diagonal subgroup of $\mathrm{PGL}_n(K)$. We denote by H_n the subgroup of $\mathrm{PGL}_n(K)$ generated by Σ_n and T_n. Then T_n is a normal subgroup of H_n and H_n is isomorphic to the semi-direct product $\mathrm{T}_n\rtimes\Sigma_n$. It is easy to prove the following lemma.

Lemma 2.2. *If $P\in\mathrm{PGL}_n(K)$ satisfies $P\mathrm{D}_nP^{-1}=\mathrm{D}_n$, then $P\in\mathrm{H}_n$.*

Since $\mathrm{PGL}_n(K)$ acts on $\mathrm{Rep}_n(m)_{ss}(K)$ by conjugation, we can extend the inclusion map $\varphi: F_n(K^m)\to\mathrm{Rep}_n(m)_{ss}(K)$ to a map $\mathrm{PGL}_n(K)\times F_n(K^m)\longrightarrow\mathrm{Rep}_n(m)_{ss}(K)$. Note that the subgroup H_n preserves the subspace $F_n(K^m)$. So this map factors through $\mathrm{PGL}_n(K)\times_{\mathrm{H}_n}F_n(K^m)$.

Theorem 2.3. *There is an isomorphism of smooth algebraic varieties with $\mathrm{PGL}_n(K)$-action*

$$\mathrm{Rep}_n(m)_{ss}(K)\cong\mathrm{PGL}_n(K)\times_{\mathrm{H}_n}F_n(K^m).$$

In particular, $\dim_K\mathrm{Rep}_n(m)_{ss}(K)=mn+n(n-1)$.

Proof. For any $A=(A_1,\dots,A_m)\in\mathrm{Rep}_n(m)_{ss}(K)$, there exists $P\in\mathrm{PGL}_n(K)$ such that $PAP^{-1}\in F_n(K^m)$. So $\mathrm{PGL}_n(K)\times_{\mathrm{H}_n}F_n(K^m)\to\mathrm{Rep}_n(m)_{ss}(K)$ is surjective. Suppose that $PUP^{-1}=QVQ^{-1}$ for $P,Q\in\mathrm{PGL}_n(K)$, $U,V\in F_n(K^m)$. Then $U=P^{-1}QVQ^{-1}P\in F_n(K^m)$. So both of U and $P^{-1}QVQ^{-1}P$ generate D_n. By Lemma 2.2, $P^{-1}Q=B\in\mathrm{H}_n$. Then we see that $(Q,V)=(PB,V)=(P,BVB^{-1})=(P,U)$ in $\mathrm{PGL}_n(K)\times_{\mathrm{H}_n}F_n(K^m)$. This completes the proof. □

Remark 2.4. There exists a scheme $\mathrm{Rep}_n(m)_{ss}$ of finite type over $\mathbb{Z}$ by [20], and $\mathrm{Rep}_n(m)_{ss}(K)$ is the associated algebraic variety.

By Theorem 2.3, we have $\mathrm{Rep}_n(m)_{ss}(K) \cong \mathrm{PGL}_n(K)/\mathrm{T}_n \times_{\Sigma_n} F_n(K^m)$. Note that $\mathrm{PGL}_n(K)/\mathrm{T}_n$ is the space of n-tuples $(l_1, \ldots, l_n)$, where l_i is a 1-dimensional subspace of K^n for $i = 1, \ldots, n$, and $\sum_{i=1}^n l_i = K^n$. Then Σ_n acts on $\mathrm{PGL}_n(K)/\mathrm{T}_n$ as permutations of lines.

We define the character variety $\mathrm{Ch}_n(m)_{ss}(K)$ of semi-simple representations as the quotient space of $\mathrm{Rep}_n(m)_{ss}(K)$ by $\mathrm{PGL}_n(K)$:

$$\mathrm{Ch}_n(m)_{ss}(K) = \mathrm{Rep}_n(m)_{ss}(K)/\mathrm{PGL}_n(K).$$

Let $\mathcal{C}_n(K^m)$ be the configuration space of distinct unordered n-points in K^m which is defined to be the quotient space $F_n(K^m)/\Sigma_n$:

$$\mathcal{C}_n(K^m) = F_n(K^m)/\Sigma_n.$$

Since $\mathrm{Ch}_n(m)_{ss}(K) = \mathrm{Rep}_n(m)_{ss}(K)/\mathrm{PGL}_n(K) \cong F_n(K^m)/\mathrm{H}_n$ and the subgroup T_n acts trivially on $F_n(K^m)$, we obtain the following corollary.

Corollary 2.5. *There is an isomorphism of smooth algebraic varieties*

$$\mathrm{Ch}_n(m)_{ss}(K) \cong \mathcal{C}_n(K^m).$$

Remark 2.6. There exists a scheme $\mathrm{Ch}_n(m)_{ss}$ of finite type over $\mathbb{Z}$ by [20], and $\mathrm{Ch}_n(m)_{ss}(K)$ is the associated algebraic variety.

2.2. **Integral cohomology groups of the moduli spaces of degree 2.** In this subsection we restrict our attention to the degree 2 representations over $\mathbb{C}$. We study the integral cohomology groups of the moduli spaces related to semi-simple representations of degree 2.

We denote by S^r the r-sphere, and by P^r the projective r-space.

Lemma 2.7. *The configuration space $F_2(\mathbb{C}^m)$ with Σ_2-action is equivariantly homotopy equivalent to S^{2m-1} with antipodal Σ_2-action. Hence $\mathcal{C}_2(\mathbb{C}^m)$ is homotopy equivalent to $\mathrm{P}^{2m-1}(\mathbb{R})$.*

Proof. We regard S^{2m-1} as the unit sphere in $\mathbb{C}^m$. We have a Σ_2-equivariant map from $F_2(\mathbb{C}^m)$ to S^{2m-1} given by $(p_1, p_2) \mapsto \frac{p_1 - p_2}{||p_1 - p_2||}$, where $|| \cdot ||$ is the standard norm in $\mathbb{C}^m$. It is easy to see that this map is a non-equivariant homotopy equivalence. Since $F_2(\mathbb{C}^m)$ and S^{2m-1} are free Σ_2-space, this map is a Σ_2-equivariant homotopy equivalence. □

By Corollary 2.5 and Lemma 2.7, we can calculate the cohomology groups of $\mathrm{Ch}_2(m)_{ss}(\mathbb{C})$.

Corollary 2.8. *We have an isomorphism of commutative graded algebras*

$$H^*(\mathrm{Ch}_2(m)_{ss}(\mathbb{C})) \cong \Lambda(s,t)/(2s, s^m, st)$$

with $|s| = 2$ and $|t| = 2m-1$.

Lemma 2.9. *The Σ_2-space $\mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_n$ is equivariantly homotopy equivalent to S^2 with antipodal Σ_2-action. Hence $\mathrm{PGL}_2(\mathbb{C})/\mathrm{H}_n$ is homotopy equivalent to $\mathrm{P}^2(\mathbb{R})$.*

Proof. Let X be the space of pairs (L_1, L_2) where L_1 and L_2 are orthogonal lines in $\mathbb{C}^2$. Then X is a Σ_2-subspace of $\mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_n$, and the inclusion $X \hookrightarrow \mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_n$ is non-equivariantly homotopy equivalent. Since $\mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_n$ and X are free Σ_2-spaces, X is Σ_2-equivariantly equivalent to $\mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_n$.

We regard $\mathrm{P}^1(\mathbb{C})$ as the space of lines in $\mathbb{C}^2$. Since the orthogonal complement $L^\perp$ of $L \in \mathrm{P}^1(\mathbb{C})$ is uniquely determined, we can identify X with $\mathrm{P}^1(\mathbb{C})$. Then the nontrivial element τ in Σ_2 acts on $\mathrm{P}^1(\mathbb{C})$ as $\tau(L) = L^\perp$ for $L \in \mathrm{P}^1(\mathbb{C})$. The space $\mathrm{P}^1(\mathbb{C})$ with this Σ_2-action can be identified with S^2 with antipodal Σ_2-action. □

Corollary 2.10. *The space $\mathrm{Rep}_2(m)_{ss}(\mathbb{C})$ is homotopy equivalent to $S^2 \times_{\Sigma_2} S^{2m-1}$.*

Proof. By Theorem 2.3, $\mathrm{Rep}_2(m)_{ss}(\mathbb{C}) \cong \mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_n \times_{\Sigma_2} F_2(\mathbb{C}^m)$. Then the corollary follows from Lemmas 2.7 and 2.9. □

Proposition 2.11. *We have an isomorphism of commutative graded algebras*

$$H^*(\mathrm{Rep}_2(1)_{ss}(\mathbb{C})) \cong \Lambda(a,b)/(2b, ab)$$

with $|a| = 1$ *and* $|b| = 3$.

Proof. By Corollary 2.10, $\mathrm{Rep}_2(1)_{ss}(\mathbb{C})$ is homotopy equivalent to $S^2 \times_{\Sigma_2} S^1$. Consider the Serre spectral sequence associated with the fibration $S^2 \to S^2 \times_{\Sigma_2} S^1 \to S^1/\Sigma_2 \cong S^1$:

$$E_2^{p,q} = H^p(\mathbb{Z}; H^q(S^2)) \Longrightarrow H^{p+q}(S^2 \times_{\Sigma_2} S^1).$$

Note that $\pi_1(S^1) \cong \mathbb{Z}$ acts nontrivially on $H^2(S^2)$. Then we have $E_2^{0,0} \cong E_2^{1,0} \cong \mathbb{Z}$, $E_2^{1,2} \cong \mathbb{Z}/2$, and $E_2^{p,q} = 0$ otherwise. Hence the spectral sequence collapses. Since there are no extension problems, we obtain the proposition. □

Theorem 2.12. *For* $m \geq 2$, *we have an isomorphism of commutative graded algebras*

$$H^*(\mathrm{Rep}_2(m)_{ss}(\mathbb{C})) \cong \Lambda(a,b)/(2a, a^2)$$

with $|a| = 2$ *and* $|b| = 2m - 1$.

Proof. By Corollary 2.10, $\mathrm{Rep}_2(m)_{ss}(\mathbb{C})$ is homotopy equivalent to $S^2 \times_{\Sigma_2} S^{2m-1}$. Consider the Serre spectral sequence associated with the fibration $S^{2m-1} \to S^2 \times_{\Sigma_2} S^{2m-1} \to S^2/\Sigma_2 \cong \mathrm{P}^2(\mathbb{R})$:

$$E_2^{p,q} = H^p(\mathrm{P}^2(\mathbb{R}); \mathcal{H}^q(S^{2m-1})) \Longrightarrow H^{p+q}(S^2 \times_{\Sigma_2} S^{2m-1}),$$

where $\mathcal{H}^q(S^{2m-1})$ is the local coefficient system determined by the action of the fundamental group $\pi_1(\mathrm{P}^2(\mathbb{R}))$ on $H^q(S^{2m-1})$. In this case $\pi_1(\mathrm{P}^2(\mathbb{R})) \cong \mathbb{Z}/2$ acts trivially on $H^{2m-1}(S^{2m-1})$. Then we have $E_2^{0,0} \cong E_2^{0,2m-1} \cong \mathbb{Z}$, $E_2^{2,0} \cong E_2^{2,2m-1} \cong \mathbb{Z}/2$, and $E_2^{p,q} = 0$ otherwise. Hence the spectral sequence collapses. Since there are no extension problems, we obtain the theorem. □

2.3. **Rational cohomology groups of the moduli spaces.** In this subsection we study the rational cohomology groups of the moduli spaces related to semi-simple representations over $\mathbb{C}$.

Recall that $\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n$ is the space of ordered n-lines $(l_1, \dots, l_n)$ in $\mathbb{C}^n$ such that $\sum_{i=1}^n l_i = \mathbb{C}^n$. We let $F_j = \sum_{i=1}^j l_j$ be the subspace of $\mathbb{C}^n$ spanned by l_i for $1 \leq i \leq j$. Then $(F_1, \dots, F_n)$ is a complete flag in $\mathbb{C}^n$. We denote by $\mathrm{Flag}(\mathbb{C}^n)$ the flag variety, which is the space of complete flags in $\mathbb{C}^n$. So we obtain a map $\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n \to \mathrm{Flag}(\mathbb{C}^n)$ of complex manifolds. Since this map is a homotopy equivalence, we obtain the following lemma.

Lemma 2.13. *The cohomology group of* $\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n$ *is given by*

$$H^*(\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n) \cong \mathbb{Z}[t_1, \dots, t_n]/(c_1, \dots, c_n),$$

where $|t_1| = \cdots = |t_n| = 2$, *and* c_i *is the* i*th elementary symmetric polynomial of* $t_1, \dots, t_n$ *for* $i = 1, \dots, n$. *The action of* Σ_n *on* $\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n$ *induces an action on* $H^*(\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n)$, *which is given by permutations of* $t_1, \dots, t_n$.

Lemma 2.14. *The rational cohomology* $H^*(\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n; \mathbb{Q})$ *is the regular representation of* Σ_n.

Proof. Since Σ_n freely acts on $\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n$, there are no fixed points. Let χ be the character of the representation defined by $H^*(\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n; \mathbb{Q})$. By the Lefschetz fixed point formula, $\chi(g) = 0$ if g is not the identity in Σ_n, and $\chi(g) = n!$ if g is the identity. Hence $H^*(\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n; \mathbb{Q})$ is the regular representation of Σ_n. □

We can easily describe the rational cohomology groups of a quotient space by a free action of a finite group. We put the following well-known lemma for the reader's convenience.

Lemma 2.15. *Let G be a finite group. Suppose that G freely acts on a space X and that the quotient map $X \to X/G$ is a principal G-bundle. Then we have an isomorphism of commutative graded algebras*

$$H^*(X/G;\mathbb{Q}) \cong H^*(X;\mathbb{Q})^G.$$

Proof. By the assumptions, we have a fibration $X \to X/G \to BG$, where BG is the classifying space of G. We consider the associated Serre spectral sequence

$$E_2^{p,q} = H^p(G; H^q(X;\mathbb{Q})) \Longrightarrow H^{p+q}(X/G;\mathbb{Q}).$$

For any G-module M over $\mathbb{Q}$, we have $H^p(G;M) = 0$ for $p > 0$ (see, for example, [1, Chapter III, Corollary 10.2]). Hence the spectral sequence collapses and we obtain that $H^p(X/G;\mathbb{Q}) \cong H^p(X;\mathbb{Q})^G$. $\square$

Theorem 2.16. *We have an isomorphism of commutative graded algebras*

$$H^*(\mathrm{Rep}_n(m)_{ss}(\mathbb{C});\mathbb{Q}) \cong \left(H^*(\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n;\mathbb{Q}) \otimes H^*(F_n(\mathbb{C}^m);\mathbb{Q})\right)^{\Sigma_n}.$$

Proof. The symmetric group Σ_n freely acts on $\mathrm{PGL}_n(\mathbb{C})/\mathrm{T}_n \times F_n(\mathbb{C}^m)$ and the quotient space $\mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_n \times_{\Sigma_n} F_n(\mathbb{C}^m)$ is isomorphic to $\mathrm{Rep}_n(m)_{ss}(\mathbb{C})$ by Theorem 2.3. Hence we obtain the theorem by Lemma 2.15. $\square$

Note that the cohomology groups of the configuration space $F_n(\mathbb{C}^m)$ is given by

$$H^*(F_n(\mathbb{C}^m)) = \Lambda(s(i,j))_{1\le i<j\le n}/I,$$

where $|s(i,j)| = 2m-1$ for $1 \le i < j \le n$ and the ideal I is generated by

$$s(i,k)s(j,k) - s(i,j)s(j,k) + s(i,j)s(i,k)$$

for $1 \le i < j < k \le n$.

Corollary 2.17. *For $n = 2$, the inclusion $F_2(\mathbb{C}^m) \hookrightarrow \mathrm{Rep}_2(m)_{ss}(\mathbb{C})$ induces an isomorphism of rational cohomology groups*

$$H^*(\mathrm{Rep}_2(m)_{ss}(\mathbb{C});\mathbb{Q}) \xrightarrow{\cong} H^*(F_2(\mathbb{C}^m);\mathbb{Q}),$$

which is an isomorphism of commutative graded algebras.

Proof. By Lemma 2.9, we see that Σ_2 non-trivially acts on $H^2(\mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_2;\mathbb{Q})$. On the other hand, Σ_2 trivially acts on $H^*(F_2(\mathbb{C}^m);\mathbb{Q})$ by Lemma 2.7. These imply that $(H^*(\mathrm{PGL}_2(\mathbb{C})/\mathrm{T}_2;\mathbb{Q}) \otimes H^*(F_2(\mathbb{C}^m);\mathbb{Q}))^{\Sigma_2} \cong H^*(F_2(\mathbb{C}^m);\mathbb{Q})$. $\square$

Remark 2.18. Corollary 2.17 also follows from Proposition 2.11 and Theorem 2.12.

Corollary 2.19. *We have*

$$\dim H^{\mathrm{even}}(\mathrm{Rep}_n(m)_{ss}(\mathbb{C});\mathbb{Q}) = \dim H^{\mathrm{odd}}(\mathrm{Rep}_n(m)_{ss}(\mathbb{C});\mathbb{Q}) = \frac{n!}{2}.$$

Proof. By induction on n, we see that $\dim H^{\mathrm{even}}(F_n(\mathbb{C}^m);\mathbb{Q}) = \dim H^{\mathrm{odd}}(F_n(\mathbb{C}^m);\mathbb{Q}) = n!/2$. Since $\mathbb{Q}[\Sigma_n] \otimes V \cong \mathbb{Q}[\Sigma_n]^{\oplus \dim V}$ as $\mathbb{Q}[\Sigma_n]$-modules for any representation V of Σ_n, we see that $H^*(\mathrm{Rep}_n(m)_{ss}(\mathbb{C});\mathbb{Q}) \cong H^*(F_n(\mathbb{C}^m);\mathbb{Q})$ as $\mathbb{Q}$-vector spaces for $* = \mathrm{even}, \mathrm{odd}$ by Lemma 2.14 and Theorem 2.16. $\square$

Lemma 2.20. *The quotient map $F_n(\mathbb{C}^m) \to \mathcal{C}_n(\mathbb{C}^m)$ induces an injection of rational cohomology groups*

$$H^*(\mathcal{C}_n(\mathbb{C}^m);\mathbb{Q}) \hookrightarrow H^*(F_n(\mathbb{C}^m);\mathbb{Q}),$$

and the image of the map is identified with the Σ_n-invariant submodule of $H^*(F_n(\mathbb{C}^m);\mathbb{Q})$. We have an isomorphism of commutative graded algebras

$$H^*(\mathcal{C}_n(\mathbb{C}^m);\mathbb{Q}) \cong \Lambda(s)$$

with $|s| = 2m-1$.

Proof. Since Σ_n freely acts on $F_n(\mathbb{C}^m)$, we obtain that $H^p(\mathcal{C}_n(\mathbb{C}^m);\mathbb{Q}) \cong H^p(F_n(\mathbb{C}^m);\mathbb{Q})^{\Sigma_n}$ by Lemma 2.15. Let $s = \sum_{i<j} s(i,j) \in H^{2m-1}(F_n(\mathbb{C}^m);\mathbb{Q})$. Then we have $H^*(F_n(\mathbb{C}^m);\mathbb{Q})^{\Sigma_n} = \Lambda(s)$ by [4, Corollary 5.2]. This completes the proof. $\square$

By Corollary 2.5, we obtain the following proposition.

Proposition 2.21. *The composition map $F_n(\mathbb{C}^m) \to \mathrm{Rep}_n(m)_{ss}(\mathbb{C}) \to \mathrm{Ch}_n(m)_{ss}(\mathbb{C})$ induces an injection of the rational cohomology groups*

$$H^*(\mathrm{Ch}_n(m)_{ss}(\mathbb{C});\mathbb{Q}) \hookrightarrow H^*(F_n(\mathbb{C}^m);\mathbb{Q}).$$

We have an isomorphism of commutative graded algebras

$$H^*(\mathrm{Ch}_n(m)_{ss}(\mathbb{C});\mathbb{Q}) \cong \Lambda(s)$$

with $|s| = 2m-1$.

3. The moduli spaces of unipotent representations of degree 2

In this section we study the moduli spaces related to unipotent representations of degree 2. We give descriptions for the moduli spaces and calculate the cohomology groups of them.

3.1. **Descriptions for the moduli spaces of unipotent representations of degree 2.** Let K be an algebraically closed field. Let N_2 be the K-subalgebra of $\mathrm{M}_2(K)$ generated by the following matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that $\dim_K \mathrm{N}_2 = 2$. Let $\mathrm{U}_2(m)(K)$ be the subspace of $(\mathrm{N}_2)^m$ given by

$$\mathrm{U}_2(m)(K) = \{(A_1,\ldots,A_m) \in (\mathrm{N}_2)^m \mid A_1,\ldots,A_m \text{ generate } \mathrm{N}_2 \text{ as a } K\text{-algebra}\}.$$

Note that $\mathrm{U}_2(m)(K)$ is an algebraic variety associated to a scheme $\mathrm{U}_2(m)$ over $\mathbb{Z}$. We define the moduli space $\mathrm{Rep}_2(m)_u(K)$ of unipotent representations of degree 2 by

$$\mathrm{Rep}_2(m)_u(K) = \left\{ (A_1,\ldots,A_m) \in (\mathrm{M}_2(K))^m \;\middle|\; \begin{array}{l} (PA_1P^{-1},\ldots,PA_mP^{-1}) \in \mathrm{U}_2(m)(K) \\ \text{for some } P \in \mathrm{PGL}_2(K) \end{array} \right\}.$$

In [20] we showed that there exists a scheme $\mathrm{Rep}_2(m)_u$ of finite type over $\mathbb{Z}[1/2]$ (or $\mathbb{Z}/2\mathbb{Z}$). Hence $\mathrm{Rep}_2(m)_u(K)$ is an algebraic variety over K associated to the scheme $\mathrm{Rep}_2(m)_u$. Note that there is a map $\mathrm{U}_2(m)(K) \to \mathrm{Rep}_2(m)_u(K)$ of algebraic varieties which is injective as a map of sets.

Let $\mathrm{B}_2(K)$ be the subgroup of $\mathrm{PGL}_2(K)$ consisting of upper triangular matrices. The group $\mathrm{B}_2(K)$ acts on $\mathrm{U}_2(m)(K)$ by conjugation. Since $\mathrm{PGL}_2(K)$ acts on $\mathrm{Rep}_2(m)_u(K)$ by conjugation, the map $\mathrm{U}_2(m)(K) \to \mathrm{Rep}_2(m)_u(K)$ extends to a map $\mathrm{PGL}_2(K) \times \mathrm{U}_2(m)(K) \to \mathrm{Rep}_2(m)_u(K)$ of algebraic varieties. This map factors through the quotient algebraic variety $\mathrm{PGL}_2(K) \times_{\mathrm{B}_2(K)} \mathrm{U}_2(m)(K)$, and hence we obtain a map

$$\mathrm{PGL}_2(K) \times_{\mathrm{B}_2(K)} \mathrm{U}_2(m)(K) \longrightarrow \mathrm{Rep}_2(m)_u(K)$$

of algebraic varieties.

It is easy to prove the following lemma .

Lemma 3.1. *If $P \in \mathrm{PGL}_2(K)$ satisfies $P\mathrm{N}_2P^{-1} = \mathrm{N}_2$, then $P \in \mathrm{B}_2(K)$.*

Theorem 3.2. *The map* $\mathrm{PGL}_2(K) \times_{\mathrm{B}_2(K)} \mathrm{U}_2(m)(K) \to \mathrm{Rep}_2(m)_u(K)$ *of algebraic varieties is a bijection*

$$\mathrm{PGL}_2(K) \times_{\mathrm{B}_2(K)} \mathrm{U}_2(m)(K) \xrightarrow{\cong} \mathrm{Rep}_2(m)_u(K)$$

as a map of sets.

Proof. For any $A = (A_1, \ldots, A_m) \in \mathrm{Rep}_2(m)_u(K)$, there exists $P \in \mathrm{PGL}_2(K)$ such that $PAP^{-1} \in \mathrm{N}_2$. So $\mathrm{PGL}_2(K) \times_{\mathrm{B}_2(K)} \mathrm{U}_2(m)(K) \to \mathrm{Rep}_2(m)_u(K)$ is surjective. Suppose that $PUP^{-1} = QVQ^{-1}$ for $P, Q \in \mathrm{PGL}_2(K)$ and $U, V \in \mathrm{U}_2(m)(K)$. Then $U = P^{-1}QVQ^{-1}P \in \mathrm{U}_2(m)(K)$. So both of U and $P^{-1}QVQ^{-1}P$ generate N_2. By Lemma 3.1, $P^{-1}Q = B \in \mathrm{B}_2(K)$. We see that $(Q, V) = (PB, V) = (P, BVB^{-1}) = (P, U)$ in $\mathrm{PGL}_2(K) \times_{\mathrm{B}_2(K)} \mathrm{U}_2(m)(K)$. This completes the proof. □

Remark 3.3. If the characteristic of K is not 2, then the map is an isomorphism of algebraic varieties. If the characteristic of K is 2, then the map induces a purely inseparable extension of degree 2 between the function fields.

We define the character variety $\mathrm{Ch}_2(m)_u(K)$ of unipotent representations of degree 2 as the quotient algebraic variety of $\mathrm{Rep}_2(m)_u(K)$ by $\mathrm{PGL}_2(K)$:

$$\mathrm{Ch}_2(m)_u(K) = \mathrm{Rep}_2(m)_u(K)/\mathrm{PGL}_2(K).$$

The map $\mathrm{PGL}_2(K) \times_{\mathrm{B}_2(K)} \mathrm{U}_2(m)(K) \to \mathrm{Rep}_2(m)_u(K)$ induces a map

$$\mathrm{U}_2(m)(K)/\mathrm{B}_2(K) \longrightarrow \mathrm{Ch}_2(m)_u(K)$$

of algebraic varieties.

Recall that for $(A_1, \ldots, A_m) \in (\mathrm{M}_2(K))^m$, we have $a_{ij} = (a_{ij}(1), \ldots, a_{ij}(m))$ for $i, j = 1, 2$, where $A_k = (a_{ij}(k))$ $(k = 1, \ldots, m)$. The map $(\mathrm{M}_2(K))^m \to (K^m)^2$ given by $(A_1, \ldots, A_m) \mapsto (a_{11}, a_{12})$ induces an isomorphism

$$\mathrm{U}_2(m)(K) \cong K^m \times (K^m - 0).$$

of algebraic varieties. By this isomorphism, we obtain an isomorphism

$$\mathrm{U}_2(m)(K)/\mathrm{B}_2(K) \cong K^m \times \mathrm{P}^{m-1}(K),$$

of algebraic varieties. This induces a map

$$K^m \times \mathrm{P}^{m-1}(K) \longrightarrow \mathrm{Ch}_2(m)_u(K)$$

of algebraic varieties.

We easily obtain the following corollary by Theorem 3.2.

Corollary 3.4. *The map* $K^m \times \mathrm{P}^{m-1}(K) \to \mathrm{Ch}_2(m)_u(K)$ *of algebraic varieties is a bijection*

$$K^m \times \mathrm{P}^{m-1}(K) \xrightarrow{\cong} \mathrm{Ch}_2(m)_u(K)$$

as a map of sets.

Remark 3.5. If the characteristic of K is not 2, then the map is an isomorphism of algebraic varieties. If the characteristic of K is 2, then the map induces a purely inseparable extension of degree 2 between the function fields.

3.2. **Cohomology groups of the moduli spaces of unipotent representations of degree 2.** In this subsection we study the integral cohomology groups of the moduli spaces of unipotent representations of degree 2 over $\mathbb{C}$. First, we treat $\mathrm{Rep}_2(1)_u(\mathbb{C})$.

Proposition 3.6. *The space* $\mathrm{Rep}_2(1)_u(\mathbb{C})$ *is homotopy equivalent to* $\mathrm{P}^3(\mathbb{R})$*. Hence we have an isomorphism of commutative graded algebras*

$$H^*(\mathrm{Rep}_2(1)_u(\mathbb{C})) \cong \Lambda(s,t)/(2s, s^2, st)$$

with $|s| = 2$ *and* $|t| = 3$*.*

Proof. By Theorem 3.2 and Remark 3.3, $\mathrm{Rep}_2(1)_u(\mathbb{C}) \cong \mathrm{PGL}_2(\mathbb{C}) \times_{\mathrm{B}_2(\mathbb{C})} \mathrm{U}_2(1)(\mathbb{C})$. Recall that $\mathrm{U}_2(1) \cong \mathbb{C} \times \mathbb{C}^\times$. Then $\mathrm{B}_2(\mathbb{C})$ acts trivially on the left factor $\mathbb{C}$ and transitively on the right factor $\mathbb{C}^\times$. Let S be the stabilizer subgroup of $\mathrm{B}_2(\mathbb{C})$ at $1 \in \mathbb{C}^\times$. Then

$$S = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}) \right\}.$$

We can write $\mathrm{U}_2(1)(\mathbb{C}) \cong \mathbb{C} \times (\mathrm{B}_2(\mathbb{C})/S)$. Hence $\mathrm{Rep}_2(1)_u(\mathbb{C}) \cong (\mathrm{PGL}_2(\mathbb{C})/S) \times \mathbb{C} \simeq \mathrm{PGL}_2(\mathbb{C})/S$. Since $S \cong \mathbb{C}$, $\mathrm{PGL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})/S$ is a homotopy equivalence. So we see that $\mathrm{Rep}_2(1)_u(\mathbb{C}) \simeq \mathrm{PGL}_2(\mathbb{C})$. It is well-known that the inclusion $\mathrm{PU}(2) \to \mathrm{PGL}_2(\mathbb{C})$ induces a homotopy equivalence $\mathrm{PU}(2) \simeq \mathrm{PGL}_2(\mathbb{C})$. Hence $\mathrm{Rep}_2(1)_u(\mathbb{C})$ is homotopy equivalent to $\mathrm{PU}(2) \cong \mathrm{P}^3(\mathbb{R})$. □

Corollary 3.7. *We have an isomorphism of commutative graded algebras*

$$H^*(\mathrm{Rep}_2(1)_u(\mathbb{C}); \mathbb{Q}) \cong \Lambda(t)$$

with $|t| = 3$*.*

For $m \geq 2$ we have the following theorem on the cohomology groups of $\mathrm{Rep}_2(m)_u(\mathbb{C})$.

Theorem 3.8. *For* $m \geq 2$*, we have an isomorphism of commutative graded algebras*

$$H^*(\mathrm{Rep}_2(m)_u(\mathbb{C})) \cong \Lambda(u,s)/(u^2)$$

with $|u| = 2$ *and* $|s| = 2m-1$*.*

Proof. By Theorem 3.2 and Remark 3.3, there is a fibre bundle $\mathrm{U}_2(m)(\mathbb{C}) \to \mathrm{Rep}_2(m)_u(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})/\mathrm{B}_2(\mathbb{C}) \cong \mathrm{P}^1(\mathbb{C})$. Since $\mathrm{U}_2(m)(\mathbb{C}) \simeq S^{2m-1}$, the associated Serre spectral sequence collapses. This completes the proof. □

Corollary 3.9. *For* $m \geq 2$*, we have an isomorphism of commutative graded algebras*

$$H^*(\mathrm{Rep}_2(m)_u(\mathbb{C}); \mathbb{Q}) \cong \Lambda(u,s)/(u^2)$$

with $|u| = 2$ *and* $|s| = 2m-1$*.*

Proof. This follows from the fact that the integral cohomology groups of $\mathrm{Rep}_2(m)_u(\mathbb{C})$ is torsion-free by Theorem 3.8. □

By Corollary 3.4 and Remark 3.5, the cohomology groups of $\mathrm{Ch}_2(m)_u(\mathbb{C})$ is given as follows.

Proposition 3.10. *We have isomorphisms of commutative graded algebras*

$$H^*(\mathrm{Ch}_2(m)_u(\mathbb{C})) \cong \mathbb{Z}[t]/(t^m)$$

and

$$H^*(\mathrm{Ch}_2(m)_u(\mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}[t]/(t^m)$$

with $|t| = 2$*.*

4. The moduli spaces of scalar representations

In this section we define the moduli spaces related to scalar representations. It is easy to give descriptions of the moduli spaces. Then we obtain the cohomology groups of them.

Let K be an algebraically closed field. We define the moduli space $\mathrm{Rep}_n(m)_{sc}(K)$ of scalar representations by

$$\mathrm{Rep}_n(m)_{sc}(K) = \{(A_1, \dots, A_m) \in (M_n(K))^m |\ \dim_K K\langle A_1, \dots, A_m\rangle = 1\},$$

where $K\langle A_1, \dots, A_m\rangle$ is the K-subalgebra of $M_n(K)$ generated by $A_1, \dots, A_m$.

Theorem 4.1. *We have an isomorphism of smooth algebraic varieties*

$$\mathrm{Rep}_n(m)_{sc}(K) \cong K^m.$$

Hence we have isomorphisms of commutative graded algebras

$$H^*(\mathrm{Rep}_n(m)_{sc}(\mathbb{C})) \cong \mathbb{Z}$$

and

$$H^*(\mathrm{Rep}_n(m)_{sc}(\mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}.$$

Proof. If $(A_1, \dots, A_m) \in \mathrm{Rep}_n(m)_{sc}(K)$, then A_i is a scalar matrix for $1 \le i \le m$. Hence $\mathrm{Rep}_n(m)_{sc}(K) \cong K^m$. □

The group $\mathrm{PGL}_n(K)$ acts on $\mathrm{Rep}_2(m)_{sc}(K)$ by conjugation. We define the character variety $\mathrm{Ch}_n(m)_{sc}(K)$ of scalar representations by the quotient space

$$\mathrm{Ch}_n(m)_{sc}(K) = \mathrm{Rep}_n(m)_{sc}(K)/\mathrm{PGL}_n(K).$$

Theorem 4.2. *We have an isomorphism of smooth algebraic varieties*

$$\mathrm{Ch}_n(m)_{sc}(K) \cong K^m.$$

Hence we have isomorphisms of commutative graded algebras

$$H^*(\mathrm{Ch}_n(m)_{sc}(\mathbb{C})) \cong \mathbb{Z}$$

and

$$H^*(\mathrm{Ch}_n(m)_{sc}(\mathbb{C}); \mathbb{Q}) \cong \mathbb{Q}$$

Proof. This follows from the fact that the action of $\mathrm{PGL}_n(K)$ on $\mathrm{Rep}_n(m)_{sc}(K)$ is trivial since $\mathrm{Rep}_n(m)_{sc}(K)$ consists of m-tuples of scalar matrices. □

Remark 4.3. There exist smooth schemes $\mathrm{Rep}_n(m)_{sc} \cong \mathbb{A}^m_{\mathbb{Z}}$ and $\mathrm{Ch}_n(m)_{sc} \cong \mathbb{A}^m_{\mathbb{Z}}$ over $\mathbb{Z}$, and $\mathrm{Rep}_n(m)_{sc}(K)$ and $\mathrm{Ch}_n(m)_{sc}(K)$ are the associated algebraic varieties.

5. Virtual Hodge polynomial of the moduli spaces

In this section we study the virtual Hodge polynomials of the moduli spaces of representations of degree 2 over $\mathbb{C}$. See, for examples, [5] for the precise definition and properties of the virtual Hodge polynomial. Also, see [21, 22] for the virtual Hodge polynomials of the moduli spaces of representations with Borel mold.

For a mixed Hodge structure (V, W, F), we denote by $a^{p,q}(V)$ the dimension of the (p,q)-component of the pure Hodge structure $\mathrm{Gr}^W_{p+q}(V)$ of weight $p+q$. For an algebraic scheme X over $\mathbb{C}$, we denote by $\mathrm{VHP}(X)$ the virtual Hodge polynomial of X:

$$\mathrm{VHP}(X) := \sum_{p,q,n} (-1)^n a^{p,q}(H^n(X;\mathbb{Q}))x^p y^q.$$

We also denote by $\mathrm{VHP}_c(X)$ the virtual Hodge polynomial of X based on the compact support cohomology. Note that if X is smooth of pure dimension m, then

$$\mathrm{VHP}_c(X) = (xy)^m \mathrm{VHP}(X)\left(x^{-1}, y^{-1}\right) \tag{5.1}$$

by the Poincaré duality. For simplicity, we set $z = xy$.

5.1. **Virtual Hodge polynomials of the moduli spaces of representations of degree 2.** In this subsection we study the virtual Hodge polynomials of the moduli spaces of representations of degree 2 over $\mathbb{C}$. In particular, we calculate the virtual Hodge polynomial of the moduli space of absolutely irreducible representations of degree 2.

Let K be an algebraically closed field. Let $\mathrm{Rep}_n(m)(K) = (\mathrm{M}_n(K))^m$ be the representation variety of degree n for the free monoid with m generators. We define the subvariety $\mathrm{Rep}_n(m)_{\mathrm{rk}h}(K)$ of $\mathrm{Rep}_n(m)(K)$ by

$$\mathrm{Rep}_n(m)_{\mathrm{rk}h}(K) = \{(A_1, \ldots, A_m) \in (\mathrm{M}_n(K)^m |\ \dim_K K\langle A_1, \ldots, A_m\rangle = h\}.$$

The representation variety $\mathrm{Rep}_n(m)_{air}(K)$ of absolutely irreducible representations is defined by

$$\mathrm{Rep}_n(m)_{air}(K) = \mathrm{Rep}_n(m)_{\mathrm{rk}n^2}(K).$$

Remark 5.1. By [18], there exists a smooth scheme $\mathrm{Rep}_n(m)_{air}$ over $\mathbb{Z}$ and $\mathrm{Rep}_n(m)_{air}(K)$ is the associated algebraic variety.

Let $\mathrm{Rep}_n(m)_B(K)$ be the representation variety of representations with Borel mold. When $n = 2$, we have $\mathrm{Rep}_2(m)_B(K) = \mathrm{Rep}_2(m)_{\mathrm{rk}3}(K)$. We calculated the virtual Hodge polynomial of $\mathrm{Rep}_n(m)_B(\mathbb{C})$ in [21] and [22].

Proposition 5.2 ([21, Proposition 7.9] and [22, Corollary 8.16])**.** *The virtual Hodge polynomial of* $\mathrm{Rep}_n(m)_B(\mathbb{C})$ *is given by*

$$\begin{aligned}
\mathrm{VHP}(\mathrm{Rep}_n(m)_B(\mathbb{C})) &= \frac{(1-z^{m-1})^{n-1}\prod_{k=1}^{n-1}(1-kz^m)\prod_{i=1}^{n}(1-z^i)}{(1-z)^n}, \\
\mathrm{VHP}_c(\mathrm{Rep}_n(m)_B(\mathbb{C})) &= \frac{z^{m(n-1)(n-2)/2}(z^m-z)^{n-1}\prod_{k=0}^{n-1}(z^m-k)\prod_{k=1}^{n}(z^k-1)}{(z-1)^n}.
\end{aligned}$$

Next we consider the virtual Hodge polynomial of $\mathrm{Rep}_2(m)_{ss}(\mathbb{C})$.

Proposition 5.3. *The virtual Hodge polynomial of* $\mathrm{Rep}_2(m)_{ss}(\mathbb{C})$ *is given by*

$$\begin{aligned}
\mathrm{VHP}(\mathrm{Rep}_2(m)_{ss}(\mathbb{C})) &= 1 - z^m, \\
\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{ss}(\mathbb{C})) &= z^{m+2}(z^m - 1).
\end{aligned}$$

Proof. By Corollary 2.17, the inclusion $F_2(\mathbb{C}^m) \hookrightarrow \mathrm{Rep}_2(m)_{ss}(\mathbb{C})$ induces an isomorphism of rational cohomology groups. Hence $\mathrm{VHP}(\mathrm{Rep}_2(m)_{ss}(\mathbb{C})) = \mathrm{VHP}(F_2(\mathbb{C}^m)) = 1 - z^m$. Since $\dim_{\mathbb{C}} \mathrm{Rep}_2(m)_{ss}(\mathbb{C}) = 2m + 2$, we have

$$\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{ss}(\mathbb{C}))(z) = z^{2m+2}\mathrm{VHP}(\mathrm{Rep}_2(m)_{ss}(\mathbb{C}))(z^{-1}).$$

Hence we obtain that $\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{ss}(\mathbb{C})) = z^{m+2}(z^m - 1)$. □

The virtual Hodge polynomial of $\mathrm{Rep}_2(m)_u(\mathbb{C})$ is given as follows.

Proposition 5.4. *The virtual Hodge polynomial of* $\mathrm{Rep}_2(m)_u(\mathbb{C})$ *is given by*

$$\begin{aligned}
\mathrm{VHP}(\mathrm{Rep}_2(m)_u(\mathbb{C})) &= (1+z)(1-z^m), \\
\mathrm{VHP}_c(\mathrm{Rep}_2(m)_u(\mathbb{C})) &= z^m(z+1)(z^m-1).
\end{aligned}$$

Proof. By Theorem 3.2 and Remark 3.3, we have the fiber bundle $\mathrm{U}_2(m)(\mathbb{C}) \to \mathrm{Rep}_2(m)_u(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})/\mathrm{B}_2(\mathbb{C})$ with respect to the Zariski topology. Note that $\mathrm{U}_2(m)(\mathbb{C}) \cong \mathbb{C}^m \times (\mathbb{C}^m - 0)$ and $\mathrm{PGL}_2(\mathbb{C})/\mathrm{B}_2(\mathbb{C}) \cong \mathrm{P}^1(\mathbb{C})$. By the property of the virtual Hodge polynomial, we obtain that

$$\mathrm{VHP}_c(\mathrm{Rep}_2(m)_u(\mathbb{C})) = \mathrm{VHP}_c(\mathrm{U}_2(m)(\mathbb{C})) \cdot \mathrm{VHP}_c(\mathrm{PGL}_2(\mathbb{C})/\mathrm{B}_2(\mathbb{C})).$$

Since $\mathrm{VHP}_c(\mathrm{U}_2(m)(\mathbb{C})) = z^m(z^m-1)$ and $\mathrm{VHP}_c(\mathrm{PGL}_2(\mathbb{C})/\mathrm{B}_2(\mathbb{C})) = 1+z$, $\mathrm{VHP}_c(\mathrm{Rep}_2(m)_u(\mathbb{C})) = z^m(z+1)(z^m-1)$. Since $\dim_{\mathbb{C}} \mathrm{Rep}_2(m)_u(\mathbb{C}) = 2m+1$, we have

$$\mathrm{VHP}(\mathrm{Rep}_2(m)_u(\mathbb{C})) = z^{2m+1}\mathrm{VHP}_c(\mathrm{Rep}_2(m)_u(\mathbb{C}))(z^{-1}).$$

Hence $\mathrm{VHP}(\mathrm{Rep}_2(m)_u(\mathbb{C})) = (1+z)(1-z^m)$. □

We consider the virtual Hodge polynomial of $\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C})$. Let U_i be the subspace of $\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C})$ consisting of $(A_1, \dots, A_m)$ such that $A_1, \dots, A_{i-1}$ are scalar matrices, and A_i is not a scalar matrix. Then we have a decomposition of $\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C})$:

$$\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C}) = U_1 \cup \cdots \cup U_m.$$

Note that $U_i \cap U_j = \emptyset$ for $i \neq j$. Since U_k is open in $\cup_{i=k}^n U_i$, we have

$$\mathrm{VHP}_c(\cup_{i=k}^n U_i) = \mathrm{VHP}_c(U_k) + \mathrm{VHP}_c(\cup_{i=k+1}^n U_i)$$

for $1 \leq k \leq n$. Hence we obtain

$$\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C})) = \sum_{i=1}^n \mathrm{VHP}_c(U_i).$$

Let I_2 be the identity matrix in $M_2(\mathbb{C})$. For $(A_1, \dots, A_m) \in U_i$, we can uniquely write $A_r = \alpha_r I_2 + \beta_r A_i$ for $r \geq i+1$, where $\alpha_r, \beta_r \in \mathbb{C}$. This implies that

$$U_i \cong \mathbb{C}^{i-1} \times (M_2(\mathbb{C}) - \mathbb{C} \cdot I_2) \times (\mathbb{C}^2)^{m-i}.$$

Then we have $\mathrm{VHP}(U_i) = 1 - z^3$ and $\mathrm{VHP}_c(U_i) = z^{i-1} \cdot (z^4 - z) \cdot z^{2(m-i)} = z^{2m-i}(z^3-1)$. Hence we obtain the following proposition.

Proposition 5.5. *The virtual Hodge polynomial of* $\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C})$ *is given by*

$$\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C})) = z^m(z^2+z+1)(z^m-1).$$

Remark 5.6. We have a decomposition

$$\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C}) = \mathrm{Rep}_2(m)_{ss}(\mathbb{C}) \cup \mathrm{Rep}_2(m)_u(\mathbb{C}),$$

where $\mathrm{Rep}_2(m)_{ss}(\mathbb{C}) \cap \mathrm{Rep}_2(m)_u(\mathbb{C}) = \emptyset$ and $\mathrm{Rep}_2(m)_{ss}(\mathbb{C})$ is open in $\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C})$. By Propositions 5.3 and 5.4, $\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{ss}(\mathbb{C})) = z^{m+2}(z^m-1)$ and $\mathrm{VHP}_c(\mathrm{Rep}_2(m)_u(\mathbb{C})) = z^m(z+1)(z^m-1)$. Hence we have

$$\begin{aligned}\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{\mathrm{rk}2}(\mathbb{C})) &= z^{m+2}(z^m-1) + z^m(z+1)(z^m-1) \\ &= z^m(z^2+z+1)(z^m-1).\end{aligned}$$

By definition, we have $\mathrm{Rep}_2(m)_{\mathrm{rk}1}(\mathbb{C}) = \mathrm{Rep}_2(m)_{\mathrm{sc}}(\mathbb{C})$. Then we obtain the following proposition by Theorem 4.1.

Proposition 5.7. *The virtual Hodge polynomial of* $\mathrm{Rep}_n(m)_{sc}(\mathbb{C})$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathrm{Rep}_n(m)_{sc}(\mathbb{C})) &= 1, \\ \mathrm{VHP}_c(\mathrm{Rep}_n(m)_{sc}(\mathbb{C})) &= z^m.\end{aligned}$$

Recall that $\mathrm{Rep}_n(m)_{air}(K) = \mathrm{Rep}_n(m)_{\mathrm{rk}n^2}(K)$ is the representation variety of absolutely irreducible representations. Since $\mathrm{Rep}_n(m)_{air}(K)$ is an open subvariety of $\mathrm{Rep}_n(m)(K) = (\mathrm{M}_n(K))^m$, $\mathrm{Rep}_n(m)_{air}(K)$ is smooth of pure dimension mn^2. When $n = 2$, we can calculate the virtual Hodge polynomial of $\mathrm{Rep}_2(m)_{air}(\mathbb{C})$ by using the above results.

Theorem 5.8 (Theorem 1.1)**.** *The virtual Hodge polynomial of* $\mathrm{Rep}_2(m)_{air}(\mathbb{C})$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathrm{Rep}_2(m)_{air}(\mathbb{C})) &= (1-z^m)(1-z^{m-1}),\\ \mathrm{VHP}_c(\mathrm{Rep}_2(m)_{air}(\mathbb{C})) &= z^{2m+1}(z^m-1)(z^{m-1}-1).\end{aligned}$$

Proof. First, we calculate $\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{air}(\mathbb{C}))$. We see that $\mathrm{VHP}_c(\mathrm{Rep}_2(m)) = z^{4m}$ since $\mathrm{Rep}_2(m)(\mathbb{C}) = M_2(\mathbb{C})^m$. We have a decomposition of $\mathrm{Rep}_2(m)(\mathbb{C})$:

$$\mathrm{Rep}_2(m)(\mathbb{C}) = \cup_{h=1}^4 \mathrm{Rep}_2(m)_{\mathrm{rk}h}(\mathbb{C}).$$

Then $\mathrm{Rep}_2(m)_{\mathrm{rk}i}(\mathbb{C}) \cap \mathrm{Rep}_2(m)_{\mathrm{rk}j}(\mathbb{C}) = \emptyset$ if $i \neq j$. Furthermore, $\mathrm{Rep}_2(m)_{\mathrm{rk}h}(\mathbb{C})$ is closed in $\cup_{i=h}^4 \mathrm{Rep}_2(m)_{\mathrm{rk}i}(\mathbb{C})$. By the additivity property of virtual Hodge polynomial,

$$\mathrm{VHP}_c(\mathrm{Rep}_2(m)(\mathbb{C})) = \sum_{h=1}^{4} \mathrm{VHP}_c(\mathrm{Rep}_2(m)_{\mathrm{rk}h}(\mathbb{C})).$$

Recall that $\mathrm{Rep}_2(m)_{\mathrm{rk}4}(\mathbb{C}) = \mathrm{Rep}_2(m)_{air}(\mathbb{C})$ and $\mathrm{Rep}_2(m)_{\mathrm{rk}3}(\mathbb{C}) = \mathrm{Rep}_2(m)_B(\mathbb{C})$. Then we can calculate $\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{air}(\mathbb{C}))$ by Propositions 5.3, 5.4, 5.7, and 5.2.

Since $\mathrm{Rep}_2(m)_{air}(\mathbb{C})$ is smooth of pure dimension $4m$, we can calculate $\mathrm{VHP}(\mathrm{Rep}_2(m)_{air}(\mathbb{C}))$ by the formula $\mathrm{VHP}(\mathrm{Rep}_2(m)_{air}(\mathbb{C})) = z^{4m}\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{air}(\mathbb{C}))(z^{-1})$. □

5.2. **Virtual Hodge polynomials of the character varieties of degree 2.** In this subsection we study the virtual Hodge polynomials of the character varieties of degree 2 over $\mathbb{C}$.

Lemma 5.9. *The virtual Hodge polynomial of the configuration space* $\mathcal{C}_n(\mathbb{C}^m)$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathcal{C}_n(\mathbb{C}^m)) &= 1-z^m,\\ \mathrm{VHP}_c(\mathcal{C}_n(\mathbb{C}^m)) &= z^{m(n-1)}(z^m-1).\end{aligned}$$

Proof. By Lemma 2.20, the quotient map $F_n(\mathbb{C}^m) \to \mathcal{C}_n(\mathbb{C}^m)$ induces an injection on the rational cohomology groups $H^*(\mathcal{C}_n(\mathbb{C}^m);\mathbb{Q}) \hookrightarrow H^*(F_n(\mathbb{C}^m);\mathbb{Q})$, and $H^*(\mathcal{C}_n(\mathbb{C}^m);\mathbb{Q}) = \Lambda(s)$ with $|s| = 2m-1$. Since the mixed Hodge structure on $H^{2m-1}(F_n(\mathbb{C}^m);\mathbb{Q})$ is pure of type (m,m) (cf. [22, Section 5.3]), we see that the virtual Hodge polynomial of $\mathcal{C}_n(\mathbb{C}^m)$ is given by $\mathrm{VHP}(\mathcal{C}_n(\mathbb{C}^m)) = 1-z^m$. Since $\mathcal{C}_n(\mathbb{C}^m)$ is smooth of dimension mn, $\mathrm{VHP}_c(\mathcal{C}_n(\mathbb{C}^m)) = z^{mn}\mathrm{VHP}(\mathcal{C}_n(\mathbb{C}^m))(z^{-1})$. Hence we obtain that $\mathrm{VHP}_c(\mathcal{C}_n(\mathbb{C}^m)) = z^{m(n-1)}(z^m-1)$. □

By Corollary 2.5, we obtain the following proposition.

Proposition 5.10. *The virtual Hodge polynomial of* $\mathrm{Ch}_n(m)_{ss}(\mathbb{C})$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathrm{Ch}_n(m)_{ss}(\mathbb{C})) &= 1-z^m,\\ \mathrm{VHP}_c(\mathrm{Ch}_n(m)_{ss}(\mathbb{C})) &= z^{m(n-1)}(z^m-1).\end{aligned}$$

By Corollary 3.4 and Remark 3.5, we obtain the following proposition.

Proposition 5.11. *The virtual Hodge polynomial of* $\mathrm{Ch}_2(m)_u(\mathbb{C})$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathrm{Ch}_2(m)_u(\mathbb{C})) &= \frac{1-z^m}{1-z},\\ \mathrm{VHP}_c(\mathrm{Ch}_2(m)_u(\mathbb{C})) &= \frac{z^m(z^m-1)}{z-1}.\end{aligned}$$

By Theorems 4.1 and 4.2, we have $\mathrm{Ch}_n(m)_{sc}(\mathbb{C}) = \mathrm{Rep}_n(m)_{sc}(\mathbb{C}) \cong \mathbb{C}^m$. Hence we obtain the following proposition.

Proposition 5.12. *The virtual Hodge polynomial of* $\mathrm{Ch}_n(m)_{sc}(\mathbb{C})$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathrm{Ch}_n(m)_{sc}(\mathbb{C})) &= 1,\\ \mathrm{VHP}_c(\mathrm{Ch}_n(m)_{sc}(\mathbb{C})) &= z^m.\end{aligned}$$

We calculated the virtual Hodge polynomial of $\mathrm{Ch}_n(m)_B(\mathbb{C})$ in [21] and [22].

Proposition 5.13 ([21, Proposition 7.8] and [22, Corollary 8.8])**.** *The virtual Hodge polynomial of* $\mathrm{Ch}_n(m)_B(\mathbb{C})$ *is given by*

$$\begin{aligned}\mathrm{VHP}(\mathrm{Ch}_n(m)_B(\mathbb{C})) &= \frac{(1-z^{m-1})^{n-1}\prod_{k=1}^{n-1}(1-kz^m)}{(1-z)^{n-1}},\\ \mathrm{VHP}_c(\mathrm{Ch}_n(m)_B(\mathbb{C})) &= \frac{z^{(m-1)(n-1)(n-2)/2}(z^{m-1}-1)^{n-1}\prod_{k=0}^{n-1}(z^m-k)}{(z-1)^{n-1}}.\end{aligned}$$

The conjugate action of matrices induces an action of $\mathrm{PGL}_n(\mathbb{C})$ on $\mathrm{Rep}_n(m)_{air}(\mathbb{C})$. The character variety $\mathrm{Ch}_n(m)_{air}(\mathbb{C})$ of absolutely irreducible representations is defined to be the quotient space

$$\mathrm{Ch}_n(m)_{air}(\mathbb{C}) = \mathrm{Rep}_n(m)_{air}(\mathbb{C})/\mathrm{PGL}_n(\mathbb{C}).$$

We let π be the quotient map

$$\pi : \mathrm{Rep}_n(m)_{air}(\mathbb{C}) \to \mathrm{Ch}_n(m)_{air}(\mathbb{C}).$$

By [18], there exists a smooth scheme $\mathrm{Ch}_n(m)_{air}$ over $\mathbb{Z}$, and $\mathrm{Ch}_n(m)_{air}(\mathbb{C})$ is the associated algebraic variety. Furthermore, the quotient map π is induced by a map

$$\pi : \mathrm{Rep}_n(m)_{air} \to \mathrm{Ch}_n(m)_{air}$$

of schemes over $\mathbb{Z}$.

To calculate the virtual Hodge polynomial of $\mathrm{Ch}_n(m)_{air}(\mathbb{C})$, we need the multiplicative property of the virtual Hodge polynomials. Let $\varphi : X \to Y$ be a map of complex algebraic varieties. Let $\underline{\mathbb{Q}}_X$ be the constant sheaf on X. We denote by $R^q\varphi_*\underline{\mathbb{Q}}_X$ the qth higher direct image of $\underline{\mathbb{Q}}_X$, and by $R^q\varphi_!\underline{\mathbb{Q}}_X$ the qth higher direct image with compact support.

Theorem 5.14 (cf. [2, Lemma 2 and Remark 2] and [6, Theorem 6.1])**.** *Let $f : X \to Y$ be a map of complex algebraic varieties, where Y is smooth and connected. We suppose that f is a locally trivial fibration with respect to the complex topology. Suppose further that $R^q\varphi_*\underline{\mathbb{Q}}_X$, respectively $R^q\varphi_!\underline{\mathbb{Q}}_X$, is a constant sheaf on Y for all q. Then we have*

$$\mathrm{VHP}(X) = \mathrm{VHP}(Y)\cdot\mathrm{VHP}(F),$$

respectively

$$\mathrm{VHP}_c(X) = \mathrm{VHP}_c(Y)\cdot\mathrm{VHP}_c(F).$$

Let us verify if π satisfies the conditions in Theorem 5.14.

Proposition 5.15. *Let $f : X \to Y$ be a principal fibre bundle with group G over a scheme S in the sense of* [17, Definition 0.10]*. In other words, (Y, f) is a geometric quotient of X by G over S satisfying*

1. *G is flat and of finite type over S,*
2. *f is a flat morphism of finite type,*
3. *$G \times_S X \to X \times_Y X$ is an isomorphism.*

If f is smooth, then f has a local trivialization with respect to the étale topology.

Proof. By [8, 17.16.3(ii)] (cf. [14, Chapter I, Propositions 3.24 and 3.26]), there exist a surjective étale morphism $h : Y' \to Y$, and a Y-morphism $g : Y' \to X$. Hence f has a local trivialization with respect to the étale topology. □

By [18, Corollary 6.4], π is a principal fibre bundle with PGL_n. Note that π is smooth because π is flat and the $\pi^{-1}(\bar{x})$ is regular for any geometric point $\bar{x}$ of $\mathrm{Ch}_n(m)_{air}$. By Proposition 5.15, we obtain the following corollary.

Corollary 5.16. *The map π is a fibre bundle with respect to the étale topology. In particular, so is π with respect to the complex topology.*

Notice that $\mathrm{Ch}_n(m)_{air}(\mathbb{C})$ is path-connected.

Lemma 5.17. *For any q, $R^q\pi_*\underline{\mathbb{Q}}$ and $R^q\pi_!\underline{\mathbb{Q}}$ are constant sheaves on $\mathrm{Ch}_n(m)_{air}(\mathbb{C})$.*

Proof. Since π is locally trivial with respect to the complex topology, $R^q\pi_*\underline{\mathbb{Q}}$ and $R^q\pi_!\underline{\mathbb{Q}}$ are locally constant. Take a base point x in $\mathrm{Ch}_2(m)_{air}(\mathbb{C})$. The fundamental group $\pi_1(\mathrm{Ch}_n(m)_{air}(\mathbb{C}), x)$ acts on the stalks $(R^q\pi_*\underline{\mathbb{Q}})_x \cong H^q(\pi^{-1}(x); \mathbb{Q})$ and $(R^q\pi_!\underline{\mathbb{Q}})_x \cong H^q_c(\pi^{-1}(x); \mathbb{Q})$ through the map $\pi_1(\mathrm{Ch}_n(m)_{air}(\mathbb{C}), x) \to \pi_0(\mathrm{PGL}_n(\mathbb{C}))$. Since $\mathrm{PGL}_n(\mathbb{C})$ is path-connected, the action is trivial. Hence the locally constant sheaves $R^q\pi_*\underline{\mathbb{Q}}$ and $R^q\pi_*\underline{\mathbb{Q}}$ are constant. □

By the above argument, we can apply Theorem 5.14. Then we obtain the following theorem.

Theorem 5.18 (Theorem 1.2)**.** *The virtual Hodge polynomial of the character variety $\mathrm{Ch}_2(m)_{air}(\mathbb{C})$ of absolutely irreducible representations is given by*

$$\begin{aligned} \mathrm{VHP}(\mathrm{Ch}_2(m)_{air}(\mathbb{C})) &= \frac{(1-z^m)(1-z^{m-1})}{1-z^2}, \\ \mathrm{VHP}_c(\mathrm{Ch}_2(m)_{air}(\mathbb{C})) &= \frac{z^{2m}(z^m-1)(z^{m-1}-1)}{z^2-1}. \end{aligned}$$

Proof. The theorem follows from Theorem 5.8. □

6. The number of absolutely irreducible representations

Let p be a prime number and let q be a power of p. We denote by $\mathbb{F}_q$ the finite field with q elements. In this section we study the number of absolutely irreducible representations of degree 2 over $\mathbb{F}_q$ for the free monoid with m generators. For a scheme X, we denote by $|X(\mathbb{F}_q)|$ the number of $\mathbb{F}_q$-valued points of X. Then $|\mathrm{Rep}_n(m)_{air}(\mathbb{F}_q)|$ is the number of absolutely irreducible representations of degree n over $\mathbb{F}_q$ for the free monoid with m generators. We show that $|\mathrm{Ch}_n(m)_{air}(\mathbb{F}_q)|$ is the number of isomorphism classes of such representations. In case $n = 2$, we show that these numbers coincide with the virtual Hodge polynomials evaluated at q.

Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ and let $F : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ be the Frobenius map given by $F(x) = x^q$. If a scheme X is defined over $\mathbb{Z}$, F induces a map $F : X(\overline{\mathbb{F}}_q) \to X(\overline{\mathbb{F}}_q)$. Since F is a topological generator of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, $|X(\mathbb{F}_q)|$ is the number of fixed points under F.

Proposition 6.1. *The number of $\mathbb{F}_q$-valued points of $\mathrm{Ch}_2(m)_{ss}$ is given by*

$$|\mathrm{Ch}_2(m)_{ss}(\mathbb{F}_q)| = q^m(q^m-1).$$

Proof. By Corollary 2.5, we have an isomorphism

$$\mathrm{Ch}_2(m)_{ss}(\overline{\mathbb{F}}_q) \cong \mathcal{C}_2(\overline{\mathbb{F}}_q{}^m).$$

Take $x = [a, b] \in \mathcal{C}_2(\overline{\mathbb{F}}_q{}^m)$. Then $F(x) = [F(a), F(b)]$. If $F(x) = x$, then (i) $F(a) = a$ and $F(b) = b$, or (ii) $F(a) = b$ and $F(b) = a$. We denote by X_1 and X_2 the subsets of $\mathrm{Ch}_2(m)_{ss}(\mathbb{F}_q)$ consisting of elements of type (i) and (ii), respectively.

In case (i), $a, b \in (\mathbb{F}_q)^m$. Since $|F_2(\mathbb{F}_q{}^m)| = q^m(q^m - 1)$ and Σ_2 freely acts on $F_2(\mathbb{F}_q{}^m)$, the number of X_1 is given by

$$|X_1| = \frac{1}{2}q^m(q^m - 1).$$

In case (ii), $a \in (\mathbb{F}_{q^2})^m$ and $b = F(a)$. Hence b is determined by a. Since $a \neq b$, $a \in (\mathbb{F}_{q^2})^m - (\mathbb{F}_q)^m$. Noticing $[a, F(a)] = [F(a), a]$, we obtain that the number of X_2 is given by

$$|X_2| = \frac{1}{2}(q^{2m} - q^m).$$

The number of $\mathbb{F}_q$-valued points of $\mathrm{Ch}_2(m)_{ss}$ is calculated as

$$|X_1| + |X_2| = q^m(q^m - 1).$$

□

Proposition 6.2. *The number of* $\mathbb{F}_q$*-valued points of* $\mathrm{Rep}_n(m)_{ss}$ *is given by*

$$|\mathrm{Rep}_n(m)_{ss}(\mathbb{F}_q)| = q^{m+2}(q^m - 1).$$

Proof. By Theorem 2.3, we have an isomorphism

$$\mathrm{Rep}_2(m)_{ss}(\overline{\mathbb{F}}_q) \cong \mathrm{PGL}_2(\overline{\mathbb{F}}_q) \times_{\mathrm{H}_2} F_2(\overline{\mathbb{F}}_q{}^m).$$

Take $x = [G; a, b] \in \mathrm{Rep}_2(m)_{ss}(\overline{\mathbb{F}}_q)$, where $G \in \mathrm{PGL}_2(\overline{\mathbb{F}}_q)$ and $(a, b) \in F_2(\overline{\mathbb{F}}_q{}^m)$. Let us regard T_2 as a closed subgroup scheme of PGL_2. If $F(x) = x$, then (i) there exists $T \in \mathrm{T}_2(\overline{\mathbb{F}}_q)$ such that $F(G) = GT$ and $F(a) = a, F(b) = b$, or (ii) there exists $T \in \mathrm{T}_2(\overline{\mathbb{F}}_q)$ such that $F(G) = GT\tau$ and $F(a) = b, F(b) = a$, where τ is the permutation matrix corresponding to the permutation $(1, 2)$. We denote by X_1 and X_2 the subsets of $\mathrm{Rep}_2(m)_{ss}(\mathbb{F}_q)$ consisting of elements of type (i) and (ii), respectively.

In case (i), $(a, b) \in F_2(\mathbb{F}_q{}^m)$. We can take $T' \in \mathrm{T}_2(\overline{\mathbb{F}}_q)$ such that $T'F(T')^{-1} = T$. Setting $G' = GT'$, we have $F(G') = G'$, and hence $G' \in \mathrm{PGL}_2(\mathbb{F}_q)$. Then $x = [G; a, b] = [G'; a, b]$. This means we can take a representative of x as $[G'; a, b]$ with $G' \in \mathrm{PGL}_2(\mathbb{F}_q)$ and $(a, b) \in F_2(\mathbb{F}_q{}^m)$. Let $[\widetilde{G}; \widetilde{a}, \widetilde{b}]$ be another representative of x with $\widetilde{G} \in \mathrm{PGL}_2(\mathbb{F}_q)$ and $(\widetilde{a}, \widetilde{b}) \in F_2(\mathbb{F}_q{}^m)$. Then there exists $H \in \mathrm{H}_2$ such that $\widetilde{G} = G'H$ and $H^{-1}(\widetilde{a}, \widetilde{b})H = (a, b)$. In particular, $H = G'^{-1}\widetilde{G} \in \mathrm{H}_2 \cap \mathrm{PGL}_2(\mathbb{F}_q)$. Since $\mathrm{H}_2 \cap \mathrm{PGL}_2(\mathbb{F}_q)$ freely acts on $\mathrm{PGL}_2(\mathbb{F}_q) \times F_2(\mathbb{F}_q{}^m)$, the number of X_1 is given by

$$\begin{aligned} |X_1| &= \frac{|\mathrm{PGL}_2(\mathbb{F}_q)| \cdot |F_2(\mathbb{F}_q{}^m)|}{|\mathrm{H}_2 \cap \mathrm{PGL}_2(\mathbb{F}_q)|} \\ &= \frac{1}{2}q^{m+1}(q^m - 1)(q + 1). \end{aligned}$$

In case (ii), $b = F(a)$ with $a \in (\mathbb{F}_{q^2})^m - (\mathbb{F}_q)^m$. We can take $T' \in \mathrm{T}_2(\overline{\mathbb{F}}_q)$ such that $T'F^2(T')^{-1} = T$. Setting $G' = GT'\tau F(T')\tau$, we have $F(G') = G'\tau$. In particular, $F^2(G') = G'$ and $G' \in \mathrm{PGL}_2(\mathbb{F}_{q^2})$. Notice that G' has the form $G' = \begin{pmatrix} 1 & 1 \\ s & s^q \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^q \end{pmatrix}$ with $s \in \mathbb{F}_{q^2} - \mathbb{F}_q, t \in \mathbb{F}_{q^2}^{\times}$. Put $S = \begin{pmatrix} 1 & 1 \\ s & s^q \end{pmatrix}$. Then $x = [S; a, F(a)]$. So any element of X_2 has a representative of this form. Since any element of X_2 has exactly two representatives of this form, the number of X_2 is given by

$$|X_2| = \frac{1}{2}q^{m+1}(q^m - 1)(q - 1).$$

The number of $\mathbb{F}_q$-valued points of $\mathrm{Rep}_2(m)_{sc}$ is calculated as

$$|X_1| + |X_2| = q^{m+2}(q^m - 1).$$

□

Proposition 6.3. *The number of $\mathbb{F}_q$-valued points of $\mathrm{Rep}_n(m)_{sc}$ and $\mathrm{Ch}_n(m)_{sc}$ are given by*

$$|\mathrm{Rep}_2(m)_{sc}(\mathbb{F}_q)| = |\mathrm{Ch}_2(m)_{sc}(\mathbb{F}_q)| = q^m.$$

Proof. The proposition follows from Theorems 4.1 and 4.2. □

In order to calculate the number of $\mathbb{F}_q$-valued points of the other moduli spaces, we need the following lemmas.

Lemma 6.4. *Let L be a field and let $\overline{L}$ be its separable closure. Let G be an algebraic group over L and let X be a scheme of finite type over L. Suppose that G acts on X over L. If $G(\overline{L})$ freely acts on $X(\overline{L})$ and the Galois cohomology $H^1(L; G)$ is trivial, then there is a bijection*

$$(X(\overline{L})/G(\overline{L}))^{\mathrm{Gal}(\overline{L}/L)} \cong X(L)/G(L).$$

Proof. We put $\mathrm{Gal} = \mathrm{Gal}(\overline{L}/L)$. Let x be an element in $X(\overline{L})$ such that the image under the quotient map $X(\overline{L}) \to X(\overline{L})/G(\overline{L})$ is invariant under the action of Gal. Since the action of $G(\overline{L})$ on $X(\overline{L})$ is free, there exists a unique $c(\sigma) \in G(\overline{L})$ such that $x^\sigma = c(\sigma)x$ for each $\sigma \in \mathrm{Gal}$. Then c is a continuous 1-cocycle for Gal with the values in $G(\overline{L})$. Since $H^1(L; G)$ is trivial, there exists $g \in G(\overline{L})$ such that $c(\sigma) = g^\sigma g^{-1}$ for all $\sigma \in \mathrm{Gal}$. Then $(g^{-1}x)^\sigma = g^{-1}x$ and hence $g^{-1}x \in X(L)$. This means the canonical map $X(L) \to (X(\overline{L})/G(\overline{L}))^{\mathrm{Gal}}$ is surjective.

Let x_1 and x_2 be elements in $X(L)$ that coincide in $(X(\overline{L})/G(\overline{L}))^{\mathrm{Gal}}$. There exists $h \in G(\overline{L})$ such that $x_1 = hx_2$. Then $x_1 = h^\sigma x_2$ for any $\sigma \in \mathrm{Gal}$. Since the action of $G(\overline{L})$ on $X(\overline{L})$ is free, $h^\sigma = h$ and hence $h \in G(L)$. Therefore, the map $X(L) \to (X(\overline{L})/G(\overline{L}))^{\mathrm{Gal}}$ induces a bijection

$$X(L)/G(L) \xrightarrow{\cong} (X(\overline{L})/G(\overline{L}))^{\mathrm{Gal}}.$$

□

Lemma 6.5 ([12], see also [26, Chapter VI, Proposition 3])**.** *The Galois cohomology $H^1(\mathbb{F}_q; G)$ is trivial for any connected algebraic group G over $\mathbb{F}_q$.*

Proposition 6.6. *The number of $\mathbb{F}_q$-valued points of $\mathrm{Rep}_2(m)_u$ is given by*

$$|\mathrm{Rep}_2(m)_u(\mathbb{F}_q)| = q^m(q^m - 1)(q + 1).$$

Proof. By Theorem 3.2, the map $\mathrm{PGL}_2 \times_{\mathrm{B}_2} \mathrm{U}_2(m) \to \mathrm{Rep}_2(m)_u$ of algebraic varieties induces a bijection

$$\mathrm{PGL}_2(\overline{\mathbb{F}}_q) \times_{\mathrm{B}_2(\overline{\mathbb{F}}_q)} \mathrm{U}_2(m)(\overline{\mathbb{F}}_q) \xrightarrow{\cong} \mathrm{Rep}_2(m)_u(\overline{\mathbb{F}}_q).$$

Since this bijection is compatible with the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on both sides, we see that

$$\mathrm{Rep}_2(m)(\mathbb{F}_q) \cong (\mathrm{PGL}_2(\overline{\mathbb{F}}_q) \times_{\mathrm{B}_2(\overline{\mathbb{F}}_q)} \mathrm{U}_2(m)(\overline{\mathbb{F}}_q))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}.$$

Notice that the action of $\mathrm{B}_2(\overline{\mathbb{F}}_q)$ on $\mathrm{PGL}_2(\overline{\mathbb{F}}_q) \times \mathrm{U}_2(m)(\overline{\mathbb{F}}_q)$ is free. By Lemmas 6.4 and 6.5, we obtain

$$|\mathrm{Rep}_2(m)_u(\mathbb{F}_q)| = \frac{|\mathrm{PGL}_2(\mathbb{F}_q)| \cdot |\mathrm{U}_2(m)(\mathbb{F}_q)|}{|\mathrm{B}_2(\mathbb{F}_q)|}.$$

The proposition follows from the fact that $\mathrm{U}_2(m)(\mathbb{F}_q) \cong {\mathbb{F}_q}^m \times ({\mathbb{F}_q}^m - 0)$. □

Proposition 6.7. *The number of $\mathbb{F}_q$-valued points of $\mathrm{Ch}_2(m)_u$ is given by*

$$|\mathrm{Ch}_2(m)_u(\mathbb{F}_q)| = \frac{q^m(q^m-1)}{q-1}.$$

Proof. By Corollary 3.4, the map $\mathrm{U}_2(m)/\mathrm{B}_2 \to \mathrm{Ch}_2(m)_u$ of algebraic varieties induces a bijection

$$(\overline{\mathbb{F}}_q)^m \times \mathrm{P}^{m-1}(\overline{\mathbb{F}}_q) \xrightarrow{\cong} \mathrm{Ch}_2(m)_u(\overline{\mathbb{F}}_q)$$

of $\overline{\mathbb{F}}_q$-valued points. This bijection is compatible with the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on both sides, and hence we obtain the following bijection

$$(\mathbb{F}_q)^m \times \mathrm{P}^{m-1}(\mathbb{F}_q) \xrightarrow{\cong} \mathrm{Ch}_2(m)_u(\mathbb{F}_q).$$

The proposition follows from the fact that $|(\mathbb{F}_q)^m \times \mathrm{P}^{m-1}(\mathbb{F}_q)| = q^m(q^m-1)/(q-1)$. □

Proposition 6.8. *The number of $\mathbb{F}_q$-valued points of $\mathrm{Rep}_n(m)_B$ is given by*

$$|\mathrm{Rep}_n(m)_B(\mathbb{F}_q)| = \frac{q^{m(n-1)(n-2)/2}(q^m-q)^{n-1}\prod_{k=0}^{n-1}(q^m-k)\prod_{k=1}^{n}(q^k-1)}{(q-1)^n}.$$

Proof. By [21, §3], we have a bijection

$$\mathrm{Rep}_n(m)_B(\overline{\mathbb{F}}_q) \cong \mathrm{PGL}_n(\overline{\mathbb{F}}_q) \times_{\mathrm{B}_n(\overline{\mathbb{F}}_q)} \mathrm{B}_n(m)_B(\overline{\mathbb{F}}_q),$$

where we regard $\mathrm{B}_n(m)_B$ as a scheme over $\mathbb{Z}$. Notice that the action of $\mathrm{B}_n(\overline{\mathbb{F}}_q)$ on $\mathrm{PGL}_n(\overline{\mathbb{F}}_q) \times \mathrm{B}_n(m)_B(\overline{\mathbb{F}}_q)$ is free. By Lemmas 6.4 and 6.5, we obtain

$$|\mathrm{Rep}_n(m)_B(\mathbb{F}_q)| = \frac{|\mathrm{PGL}_n(\mathbb{F}_q)| \cdot |\mathrm{B}_n(m)_B(\mathbb{F}_q)|}{|\mathrm{B}_n(\mathbb{F}_q)|}$$

The proposition follows from the fact that

$$|B_n(m)_B(\mathbb{F}_q)| = q^{m(n-1)(n-2)/2}(q^m-q)^{n-1}\prod_{k=0}^{n-1}(q^m-k).$$

□

Proposition 6.9. *The number of $\mathbb{F}_q$-valued points of $\mathrm{Ch}_n(m)_B$ is given by*

$$|\mathrm{Ch}_n(m)_B(\mathbb{F}_q)| = \frac{(q^{m-1}-1)^{n-1}q^{(m-1)(n-1)(n-2)/2}\prod_{k=0}^{n-1}(q^m-k)}{(q-1)^{n-1}}.$$

Proof. By definition, there is a bijection

$$\mathrm{Ch}_n(m)_B(\overline{\mathbb{F}}_q) \cong \mathrm{Rep}_n(m)_B(\overline{\mathbb{F}}_q)/\mathrm{PGL}_n(\overline{\mathbb{F}}_q)$$

This implies that there is a bijection

$$\mathrm{Ch}_n(m)_B(\overline{\mathbb{F}}_q) \cong \mathrm{B}_n(m)_B(\overline{\mathbb{F}}_q)/\mathrm{B}_n(\overline{\mathbb{F}}_q)$$

Note that $\mathrm{B}_n(\overline{\mathbb{F}}_q)$ freely acts on $\mathrm{B}_2(m)_B(\overline{\mathbb{F}}_q)$. By Lemmas 6.4 and 6.5, we obtain

$$|\mathrm{Ch}_n(m)_B(\mathbb{F}_q)| = \frac{|\mathrm{B}_n(m)_B(\mathbb{F}_q)|}{|\mathrm{B}_n(\mathbb{F}_q)|}.$$

The proposition is easily obtained from this. □

Theorem 6.10 (cf. Theorem 1.3)**.** *The number of absolutely irreducible representations of degree* 2 *over* $\mathbb{F}_q$ *for the free monoid with* m *generators coincides with the virtual Hodge polynomial evaluated at* q *so that*

$$\begin{aligned} |\mathrm{Rep}_2(m)_{air}(\mathbb{F}_q)| &= \mathrm{VHP}_c(\mathrm{Rep}_2(m)_{air}(\mathbb{C}))(q) \\ &= q^{2m+1}(q^m-1)(q^{m-1}-1). \end{aligned}$$

Proof. By definition, we have a decomposition

$$\mathrm{M}_2(\overline{\mathbb{F}}_q)^m = \coprod_{*=sc,ss,u,B,air} \mathrm{Rep}_2(m)_*(\overline{\mathbb{F}}_q).$$

This decomposition is compatible with the map F. Taking the fixed points of F, we obtain a decomposition of $\mathrm{M}_2(\mathbb{F}_q)^m$:

$$\mathrm{M}_2(\mathbb{F}_q)^m = \coprod_{*=sc,ss,u,B,air} \mathrm{Rep}_2(m)_*(\mathbb{F}_q).$$

Hence

$$|\mathrm{Rep}_2(m)_{air}(\mathbb{F}_q)| = q^{4m} - \sum_{*=sc,ss,u,B} |\mathrm{Rep}_2(m)_*(\mathbb{F}_q)|.$$

The theorem follows from Propositions 6.2, 6.3, 6.6, and 6.8. □

Theorem 6.11 (cf. Theorem 1.3)**.** *The number of isomorphism classes of absolutely irreducible representations of degree* n *over* $\mathbb{F}_q$ *for the free monoid with* m *generators is*

$$|\mathrm{Ch}_n(m)_{air}(\mathbb{F}_q)| = \frac{|\mathrm{Rep}_n(m)_{air}(\mathbb{F}_q)|}{|\mathrm{PGL}_n(\mathbb{F}_q)|}.$$

In case $n = 2$*, the number coincides with the virtual Hodge polynomial evaluated at* q *so that*

$$\begin{aligned} |\mathrm{Ch}_2(m)_{air}(\mathbb{F}_q)| &= \mathrm{VHP}_c(\mathrm{Ch}_2(m)_{air}(\mathbb{C}))(q) \\ &= \frac{q^{2m}(q^m-1)(q^{m-1}-1)}{q^2-1}. \end{aligned}$$

Proof. Since $\mathrm{Ch}_n(m)_{air}$ is the geometric quotient of $\mathrm{Rep}_n(m)_{air}$ by PGL_n, there is a bijection

$$\mathrm{Ch}_n(m)_{air}(\overline{\mathbb{F}}_q) \cong \mathrm{Rep}_n(m)_{air}(\overline{\mathbb{F}}_q)/\mathrm{PGL}_n(\overline{\mathbb{F}}_q).$$

Notice that $\mathrm{PGL}_n(\overline{\mathbb{F}}_q)$ freely acts on $\mathrm{Rep}_n(m)_{air}(\overline{\mathbb{F}}_q)$. By Lemmas 6.4 and 6.5, we obtain

$$|\mathrm{Ch}_n(m)_{air}(\mathbb{F}_q)| = \frac{|\mathrm{Rep}_n(m)_{air}(\mathbb{F}_q)|}{|\mathrm{PGL}_n(\mathbb{F}_q)|}.$$

This shows that the number of isomorphism classes of absolutely irreducible representations over $\mathbb{F}_q$ is $|\mathrm{Ch}_n(m)_{air}(\mathbb{F}_q)|$. When $n = 2$, we can calculate the number by Theorem 6.10. □

Remark 6.12. Let X be a separated scheme of finite type over $\mathbb{Z}$. If there exists a polynomial $P_X(t) \in \mathbb{Z}[t]$ such that $|X(\mathbb{F}_q)| = P_X(q)$ for all finite fields $\mathbb{F}_q$, then $\mathrm{VHP}_c(X)$ is a polynomial of $z = xy$, and

$$\mathrm{VHP}_c(X)(z) = P_X(z).$$

See [9, §6] for more details.

Remark 6.13. Let S_m be the quiver with one vertex and m edge loops. The path algebra of S_m over a field k is the free algebra $k\langle X_1, X_2, \ldots, X_m\rangle$. Hence the representations of the quiver S_m are the same things as those of the free monoid with m generators. Let $\mathrm{AIR}_{S_m}(n,q)$ be the number of isomorphism classes of n-dimensional absolutely irreducible representations of S_m over $\mathbb{F}_q$. By Theorem 6.11, we have

$$\mathrm{AIR}_{S_m}(2,q) = \frac{q^{2m}(q^m-1)(q^{m-1}-1)}{q^2-1}.$$

Let $\mathrm{AID}_{S_m}(n,q)$ be the number of isomorphism classes of n-dimensional absolutely indecomposable representations of S_m over $\mathbb{F}_q$. Using [10, Theorem 4.6], we can calculate $\mathrm{AID}_{S_m}(2,q)$ as

$$\mathrm{AID}_{S_m}(2,q) = \frac{q^{2m-1}(q^{2m}-1)}{q^2-1}.$$

We can verify that this number is equal to the sum

$$|\mathrm{Ch}_2(m)_u(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_B(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_{air}(\mathbb{F}_q)|.$$

Let $A_n(m)$ be the affine ring of $\mathrm{Rep}_n(m)$. Let $A_n(m)^{\mathrm{PGL}_n}$ be the PGL_n-invariant ring of $A_n(m)$. We set $\mathrm{Ch}_n(m) := \mathrm{Spec} A_n(m)^{\mathrm{PGL}_n}$. By [27, Theorem 3], the set of $\overline{\mathbb{F}}_q$-valued points of $\mathrm{Ch}_n(m)$ consists of the closed PGL_n-orbits in $\mathrm{Rep}_n(m)$. In particular, when $n=2$, we have a decomposition

$$\mathrm{Ch}_2(m)(\overline{\mathbb{F}}_q) = \mathrm{Ch}_2(m)_{air}(\overline{\mathbb{F}}_q) \coprod \mathrm{Ch}_2(m)_{ss}(\overline{\mathbb{F}}_q) \coprod \mathrm{Ch}_2(m)_{sc}(\overline{\mathbb{F}}_q)$$

of $\overline{\mathbb{F}}_q$-valued points. This implies the following main theorem:

Theorem 6.14 (Theorem 1.4)**.** *The number of $\mathbb{F}_q$-valued points of $\mathrm{Ch}_2(m)$ is given by*

$$\begin{aligned} |\mathrm{Ch}_2(m)(\mathbb{F}_q)| &= |\mathrm{Ch}_2(m)_{\mathrm{air}}(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_{ss}(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_{sc}(\mathbb{F}_q)| \\ &= \frac{q^{2m+2}(q^{2m-3}-q^{m-2}-q^{m-3}+1)}{q^2-1}. \end{aligned}$$

In particular, the virtual Hodge polynomial of $\mathrm{Ch}_2(m)$ is given by

$$\mathrm{VHP}_c(\mathrm{Ch}_2(m))(z) = \frac{z^{2m+2}(z^{2m-3}-z^{m-2}-z^{m-3}+1)}{z^2-1}.$$

Remark 6.15. The Weil zeta functions of $\mathrm{Rep}_2(m)_{air}$, $\mathrm{Ch}_2(m)_{air}$, and $\mathrm{Ch}_2(m)$ are given by

$$Z(\mathrm{Rep}_2(m)_{air}, q, t) := \exp\left(\sum_{n=1}^{\infty} \frac{|\mathrm{Rep}_2(m)_{air}(\mathbb{F}_{q^n})|}{n} t^n\right) = \frac{(1-q^{3m+1}t)(1-q^{3m}t)}{(1-q^{4m}t)(1-q^{2m+1}t)},$$

$$Z(\mathrm{Ch}_2(m)_{air}, q, t) := \exp\left(\sum_{n=1}^{\infty} \frac{|\mathrm{Ch}_2(m)_{air}(\mathbb{F}_{q^n})|}{n} t^n\right) = \frac{\prod_{i=1}^{[\frac{m}{2}]}(1-q^{2m+2i-2}t)}{\prod_{i=1}^{[\frac{m}{2}]}(1-q^{4m-2i-1}t)},$$

$$Z(\mathrm{Ch}_2(m), q, t) := \exp\left(\sum_{n=1}^{\infty} \frac{|\mathrm{Ch}_2(m)(\mathbb{F}_{q^n})|}{n} t^n\right) = \frac{Z(\mathrm{Ch}_2(m)_{air}, q, t)}{1-q^{2m}t}.$$

The Hasse-Weil zeta functions of $\mathrm{Rep}_2(m)_{air}(\mathbb{C}), \mathrm{Ch}_2(m)_{air}(\mathbb{C})$, and $\mathrm{Ch}_2(m)$ are given by

$$\begin{aligned}
\zeta(\mathrm{Rep}_2(m)_{air}, s) &:= \prod_p Z(\mathrm{Rep}_2(m)_{air}, p, p^{-s}) = \frac{\zeta(s-4m)\zeta(s-2m-1)}{\zeta(s-3m-1)\zeta(s-3m)}, \\
\zeta(\mathrm{Ch}_2(m)_{air}, s) &:= \prod_p Z(\mathrm{Ch}_2(m)_{air}, p, p^{-s}) = \frac{\prod_{i=1}^{[\frac{m}{2}]} \zeta(s-4m+2i+1)}{\prod_{i=1}^{[\frac{m}{2}]} \zeta(s-2m-2i+2)}, \\
\zeta(\mathrm{Ch}_2(m), s) &:= \prod_p Z(\mathrm{Ch}_2(m), p, p^{-s}) = \zeta(\mathrm{Ch}_2(m)_{air}, s)\zeta(s-2m),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function. The completions of these zeta functions are defined as

$$\begin{aligned}
\hat{\zeta}(\mathrm{Rep}_2(m)_{air}, s) &:= \frac{\hat{\zeta}(s-4m)\hat{\zeta}(s-2m-1)}{\hat{\zeta}(s-3m-1)\hat{\zeta}(s-3m)}, \\
\hat{\zeta}(\mathrm{Ch}_2(m)_{air}, s) &:= \frac{\prod_{i=1}^{[\frac{m}{2}]} \hat{\zeta}(s-4m+2i+1)}{\prod_{i=1}^{[\frac{m}{2}]} \hat{\zeta}(s-2m-2i+2)},
\end{aligned}$$

where $\hat{\zeta}(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completion of the Riemann zeta function. Since $\hat{\zeta}(1-s) = \hat{\zeta}(s)$, we have the following functional equations

$$\begin{aligned}
\hat{\zeta}(\mathrm{Rep}_2(m)_{air}, 6m+2-s) &= \hat{\zeta}(\mathrm{Rep}_2(m)_{air}, s), \\
\hat{\zeta}(\mathrm{Ch}_2(m)_{air}, 6m-2-s) &= \hat{\zeta}(\mathrm{Ch}_2(m)_{air}, s)^{-1}.
\end{aligned}$$

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