THICK REPRESENTATIONS AND DENSE REPRESENTATIONS I

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Abstract. We introduce special classes of irreducible representations of groups: thick representations and dense representations. Denseness implies thickness, and thickness implies irreducibility. We show that absolute thickness and absolute denseness are open conditions for representations. Thereby, we can construct the moduli schemes of absolutely thick representations and absolutely dense representations. We also describe several results and several examples on thick representations for developing a theory of thick representations.

1. Introduction

We deal with special classes of irreducible representations of groups. First, we introduce the notion of thick representations. Let G be a group. Let V be an n-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-thick* if for any subspaces V_1 and V_2 of V with $\dim V_1 = m$ and $\dim V_2 = n - m$ there exists $g \in G$ such that $(\rho(g)V_1) \oplus V_2 = V$. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *thick* if ρ is m-thick for each $0 < m < n$ (Definition 2.1).

It may be expected that any irreducible representation is thick. Indeed, each irreducible representation of dimension at most 3 is thick. However, it is not true for the case of dimension n for $n \geq 4$. For example, the standard 4-dimensional representation $\mathbb{C}^4$ of $\mathrm{SO}_4(\mathbb{C})$ is not thick (Proposition 6.10). Hence it is a natural question when irreducible representations of dimension n for $n \geq 4$ are thick.

Next, we introduce another type of irreducible representations. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-dense* if the induced representation $(\wedge^m \rho) : G \to \mathrm{GL}(\wedge^m V)$ is irreducible. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *dense* if ρ is m-dense for each $0 < m < n$ (Definition 2.3). We can prove that denseness implies thickness and that thickness implies irreducibility (Corollary 2.8). For example, the standard representation $\mathbb{C}^n$ of $\mathrm{GL}_n(\mathbb{C})$ is dense, and hence thick.

The reason why we call such irreducible representations "thick" or "dense" is because the image of $\rho : G \to \mathrm{GL}(V)$ is thick or dense in $\mathrm{GL}(V)$, respectively. We imagine that if the image $\rho(G)$ gets larger in $\mathrm{GL}(V)$, then ρ may become thick or

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dense. Our purpose is to develop a theory of thick representations. Thickness is a simple, natural and essential concept in representation theory. In the case of finite-dimensional representations of complex simple Lie groups, thick representations are equivalent to weight multiplicity free representations whose weight poset is a totally ordered set ([4, Theorem 1.1]). By this result, we will classify thick representations for complex simple Lie groups in [4]. This is one of the characterization of weight multiplicity free representations whose weight poset is a totally ordered set.

We will divide "Thick representations and dense representations" into two parts: Part I and Part II, because it will be long. In Part I, we introduce thickness, denseness, realizable subspaces, and other notions on irreducible representations. We show basic results on thick representations and dense representations. For describing thickness, we introduce "realizable subspaces". We say that a subspace W of $\wedge^m V$ is *realizable* if there exist $v_1, v_2, \dots, v_m \in V$ such that $0 \neq v_1 \wedge v_2 \wedge \cdots \wedge v_m \in W$ (Definition 2.10). The notion of realizable subspaces is essential for describing criteria of thickness and the moduli of absolutely thick representations. Roughly speaking, thickness lives not in the world that linear algebra controls, but in the world that Grassmann algebra (or variety) controls. "Realizable subspaces" is one of keyphrases in Grassmann algebra.

The main theorem of Part I is the following:

Theorem 1.1 (Theorem 3.9)**.** *Let* $\mathrm{Rep}_n(G)$ *be the representation variety of degree* n *for a group* G *over* $\mathbb{Z}$*. For* $0 < m < n$*, the absolutely* m*-thick representations in* $\mathrm{Rep}_n(G)$ *form an open subscheme of* $\mathrm{Rep}_n(G)$*. In particular, the absolutely thick representations in* $\mathrm{Rep}_n(G)$ *form an open subscheme of* $\mathrm{Rep}_n(G)$*.*

Here we say that a representation $\rho : G \to \mathrm{GL}(V)$ is *absolutely* m*-thick* (resp. *absolutely thick*) if $\rho \otimes_k \overline{k} : G \to \mathrm{GL}(V \otimes_k \overline{k})$ is m-thick (resp. thick) for an algebraic closure $\overline{k}$ of k. As a corollary of the main theorem, we can construct the moduli of absolutely thick representations (Theorems 3.10).

In Part II, we will introduce (i, j)-thickness, (i, j)-denseness, and m-irreducibility as generalizations of m-thickness, m-denseness, and irreducibility, respectively. We will also describe the moduli of 4-dimensional non-thick absolutely irreducible representations of the free group F_2 of rank 2.

The organization of this article is as follows: In §2, we introduce the notions of thickness and denseness. We describe fundamental properties of thickness and denseness, and a criterion for thickness. In §3, we state the main theorem and prove the existence of the moduli schemes of absolutely thick representations and of absolutely dense representations. In §4, we investigate several results on realizable subspaces. We define the r-number $r(\wedge^m(n))$ and calculate them for small m and n. In §5, we describe useful criteria for thickness of 4-dimensional and 5-dimensional representations. In §6, we introduce several examples of thick representations and dense representations for Lie groups.

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2. m-THICKNESS AND m-DENSENESS

In this section, we introduce thickness and denseness. We describe fundamental properties of thickness and denseness, and a criterion for thickness. Proposition 2.11 is useful for verifying thickness of representations.

Definition 2.1. Let G be a group. Let V be an n-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-thick* if for any subspaces V_1 and V_2 of V with $\dim V_1 = m$ and $\dim V_2 = n - m$ there exists $g \in G$ such that $(\rho(g)V_1) \oplus V_2 = V$ (or equivalently, $(\rho(g)V_1) \cap V_2 = \{0\}$). We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *thick* if ρ is m-thick for each $0 < m < n$.

Remark 2.2. From the definition, any n-dimensional representations ρ are always 0-thick and n-thick. In particular, ρ is thick if and only if ρ is m-thick for each $0 \le m \le n$.

Definition 2.3. Let G be a group. Let V be an n-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-dense* if the induced representation $(\wedge^m \rho) : G \to \mathrm{GL}(\wedge^m V)$ is irreducible. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *dense* if ρ is m-dense for each $0 < m < n$.

Remark 2.4. For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$ over a field k, ρ is always 0-dense and n-dense because $\wedge^0 V \cong k$ and $\wedge^n V \cong k$. In particular, ρ is dense if and only if ρ is m-dense for each $0 \le m \le n$.

Lemma 2.5. *Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation of a group G. For positive integers i and j with $i + j = n$, let us consider the G-equivariant perfect pairing $\wedge^i V \otimes \wedge^j V \xrightarrow{\wedge} \wedge^n V \cong k$. For a G-invariant subspace W of $\wedge^i V$, put $W^\perp := \{y \in \wedge^j V \mid x \wedge y = 0$ for any $x \in W\}$. Then $W^\perp$ is a G-invariant subspace of $\wedge^j V$. In particular, $\wedge^i V$ is irreducible if and only if so is $\wedge^j V$.*

Proof. For $y \in W^\perp$, we have $x \wedge gy = g(g^{-1}x \wedge y) = 0$ for $x \in W$ and $g \in G$. Hence $W^\perp$ is G-invariant. The correspondence $W \mapsto W^\perp$ gives a bijection between the G-invariant subspaces of $\wedge^i V$ and $\wedge^j V$. Therefore $\wedge^i V$ is irreducible if and only if so is $\wedge^j V$. □

Proposition 2.6. *Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation of a group G. For each $0 < m < n$, ρ is m-thick (resp. m-dense) if and only if ρ is $(n - m)$-thick (resp. $(n - m)$-dense).*

Proof. It is obvious that m-thickness and $(n - m)$-thickness are equivalent. By using Lemma 2.5, we see that m-denseness and $(n-m)$-denseness are equivalent. □

Proposition 2.7. *For any n-dimensional representations $\rho : G \to \mathrm{GL}(V)$, the following implications hold for $0 < m < n$:*

$$\begin{array}{ccccc} m\text{-}dense & \Longrightarrow & m\text{-}thick & & \\ & & \Downarrow & & \\ 1\text{-}dense & \Longleftrightarrow & 1\text{-}thick & \Longleftrightarrow & irreducible. \end{array} \tag{1}$$

Proof. It suffices to prove "m-dense $\Rightarrow$ m-thick", "irreducible $\Rightarrow$ 1-dense", and "m-thick $\Rightarrow$ irreducible". First, we show "m-dense $\Rightarrow$ m-thick". Assume that $\rho : G \to \mathrm{GL}(V)$ is m-dense. Let V_1 and V_2 be vector subspaces of V with $\dim V_1 = m$ and $\dim V_2 = n - m$. The canonical homomorphism $\wedge^m V \otimes \wedge^{n-m} V \to \wedge^n V \cong k$ is a perfect pairing and G-equivariant. Let us take a basis $\{e_1, e_2, \dots, e_m\}$ of V_1 and a basis of $\{f_1, f_2, \dots, f_{n-m}\}$ of V_2. Because of irreducibility of $\wedge^m V$, the vectors $\{(\wedge^m \rho)(g)(e_1 \wedge e_2 \wedge \cdots \wedge e_m) \mid g \in G\}$ span the vector space $\wedge^m V$. Hence there exists $g \in G$ such that $(\wedge^m \rho)(g)(e_1 \wedge e_2 \wedge \cdots \wedge e_m) \wedge (f_1 \wedge f_2 \wedge \cdots \wedge f_{n-m}) \neq 0$. This implies that $(\rho(g)V_1) \oplus V_2 = V$. Therefore ρ is m-thick.

Next, we show "irreducible $\Rightarrow$ 1-dense". This follows easily from the definition. Finally, we show "m-thick $\Rightarrow$ irreducible". Assume that ρ is not irreducible. There exists a non-trivial G-invariant subspace V' of V. Set $\ell := \dim V'$. Then we only need to choose suitable subspaces V_1, V_2 of V such that $\dim V_1 = m$, $\dim V_2 = n-m$ and $(\rho(g)V_1) + V_2 \neq V$ for any $g \in G$. This implies ρ is not m-thick, which completes the proof. For the proof, we consider the following three cases: $\ell \leq \min(m, n-m)$, $\ell \geq \max(m, n-m)$, and $\min(m, n-m) < \ell < \max(m, n-m)$. If $\ell \leq \min(m, n-m)$, then let us take subspaces V_1, V_2 of V such that $V' \subseteq V_1$ and $V' \subseteq V_2$. Since $\rho(g)V_1 \supseteq \rho(g)V' = V'$ and $V_2 \supseteq V'$, $(\rho(g)V_1) \cap V_2 \supseteq V' \neq 0$ for each $g \in G$. In this case, $(\rho(g)V_1) + V_2 \neq V$ for any $g \in G$, and hence ρ can not be m-thick. If $\ell \geq \max(m, n-m)$, then let us take subspaces V_1, V_2 of V such that $V_1 \subseteq V'$ and $V_2 \subseteq V'$. Since $(\rho(g)V_1) + V_2 \subseteq V' \neq V$, ρ is not m-thick. In the case $\min(m, n-m) < \ell < \max(m, n-m)$, we may assume that $n - m \geq m$ because m-thickness and $(n-m)$-thickness are equivalent. Then let us take subspaces V_1, V_2 of V such that $V_1 \subseteq V' \subseteq V_2$. Since $\rho(g)V_1 \subseteq V' \subseteq V_2$, $(\rho(g)V_1) \cap V_2 = \rho(g)V_1 \neq 0$. Hence $(\rho(g)V_1) + V_2 \neq V$ for each $g \in G$, which implies ρ is not m-thick. □

Corollary 2.8. *For any finite-dimensional representation of a group G, the following implications hold:*

$$dense \Rightarrow thick \Rightarrow irreducible. \tag{2}$$

Corollary 2.9. *Assume that $\dim V \leq 3$. Then for a representation $\rho : G \to \mathrm{GL}(V)$, the following implications hold:*

$$dense \Leftrightarrow thick \Leftrightarrow irreducible. \tag{3}$$

Proof. The statement follows from that the three conditions above are equivalent to 1-dense (1-thick, or irreducible) when $\dim V \leq 3$. □

Now we consider a criterion for a representation to be m-thick. This criterion of m-thickness will be used for describing the moduli of absolutely thick representations. Before introducing it, we need the following definition.

Definition 2.10. Let V be a finite-dimensional vector space over a field k. For a vector subspace $W \subseteq \wedge^m V$, we say that W is *realizable* over k if there exist $v_1, v_2, \dots, v_m \in V$ such that $0 \neq v_1 \wedge v_2 \wedge \dots \wedge v_m \in W$. For an m-dimensional subspace V' of V with $0 < m < n$, we can consider a point $[\wedge^m V'] \in \mathbb{P}_*(\wedge^m V)$. In the sequel, we identify $[\wedge^m V']$ with a non-zero vector $\wedge^m V' \in \wedge^m V$ (which is determined by $[\wedge^m V']$ up to scalar) for simplicity. It is obvious that W is realizable if and only if W contains a non-zero vector $\wedge^m V'$ obtained by an m-dimensional subspace V' of V over k.

The following proposition gives a criterion of m-thickness.

Proposition 2.11. *Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation of a group G. For $0 < m < n$, ρ is not m-thick if and only if there exist G-invariant realizable vector subspaces $W_1 \subset \wedge^m V$ and $W_2 \subset \wedge^{n-m} V$ such that $W_1^{\perp} = W_2$.*

Proof. Suppose that ρ is not m-thick. Then there exist vector subspaces V_1, V_2 of V with $\dim V_1 = m$ and $\dim V_2 = n - m$ such that $(\rho(g)V_1) + V_2 \neq V$ for any $g \in G$. Let us consider the vector $\wedge^m V_1 \in \wedge^m V$ determined by V_1 up to scalar multiplication. The condition implies that vectors $\{(\wedge^m \rho)(g)(\wedge^m V_1) \mid g \in G\}$ span a non-trivial G-invariant subspace $W_1 \subset \wedge^m V$. Of course, W_1 is realizable. Set $W_2 := W_1^{\perp} \subset \wedge^{n-m} V$. Note that $\wedge^{n-m} V_2 \in W_2$. The subspace W_2 is a non-trivial G-invariant realizable subspace. Hence we have proved the "only if" part.

Conversely, suppose that there exist G-invariant realizable vector subspaces $W_1 \subseteq \wedge^m V$ and $W_2 \subseteq \wedge^{n-m} V$ such that $W_1^{\perp} = W_2$. Since W_1 and W_2 are realizable, there exist an m-dimensional subspace $V_1 \subseteq V$ and an $(n-m)$-dimensional subspace $V_2 \subseteq V$ such that $\wedge^m V_1 \in W_1$ and $\wedge^{n-m} V_2 \in W_2$. For each $g \in G$, the vector $(\wedge^m \rho)(g)(\wedge^m V_1)$ is contained in W_1, and hence $(\wedge^m \rho)(g)(\wedge^m V_1) \wedge (\wedge^{n-m} V_2) = 0$. This implies that $(\rho(g)V_1) + V_2 \neq V$ for each $g \in G$. Therefore ρ is not m-thick. □

Remark 2.12. Furthermore, we also see that ρ is not m-thick if and only if there exist a non-zero G-invariant realizable subspace $W_1 \subset \wedge^m V$ and an $(n-m)$-dimensional subspace V' of V such that $\wedge^{n-m} V' \in W_1^{\perp}$.

Let us define absolute thickness and absolute denseness. We will construct the moduli spaces of absolutely thick representations and absolutely dense representations in the next section.

Definition 2.13. Let G be a group. Let V be an n-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *absolutely m-thick* if $\rho \otimes \overline{k} : G \to \mathrm{GL}(V \otimes \overline{k})$ is m-thick, where $\overline{k}$ is an algebraic closure of k. We also say that ρ is *absolutely thick* if ρ is absolutely m-thick for each $0 < m < n$.

Definition 2.14. Let G be a group. Let V be an n-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *absolutely m-dense* if $\rho \otimes \overline{k} : G \to \mathrm{GL}(V \otimes \overline{k})$ is m-dense, where $\overline{k}$ is an algebraic closure of k. We also say that ρ is *absolutely dense* if ρ is absolutely m-dense for each $0 < m < n$.

Remark 2.15. Let K be an extension field of k. If $\rho \otimes_k K : G \to \mathrm{GL}(V \otimes_k K)$ is m-thick (resp. m-dense), then ρ is also m-thick (resp. m-dense). In particular, if ρ is absolutely m-thick (resp. absolutely m-dense), then ρ is m-thick (resp. m-dense).

Proposition 2.16. *For an n-dimensional group representation $\rho : G \to \mathrm{GL}(V)$, the following conditions are equivalent:*

(1) *ρ is absolutely m-dense, in other words, $(\wedge^m \rho) \otimes_k \overline{k} : G \to \mathrm{GL}(\wedge^m V \otimes_k \overline{k})$ is irreducible, where $\overline{k}$ is an algebraically closure of k.*
(2) *$(\wedge^m \rho) \otimes_k K : G \to \mathrm{GL}(\wedge^m V \otimes_k K)$ is irreducible for some algebraically closed field K containing k.*
(3) *$(\wedge^m \rho) \otimes_k K : G \to \mathrm{GL}(\wedge^m V \otimes_k K)$ is irreducible for any algebraically closed field K containing k.*

Proof. The statement follows from that all conditions above are equivalent to the condition that $\wedge^m \rho$ is absolutely irreducible. □

In Theorem 3.8, we will obtain the same result on absolute m-thickness as Proposition 2.16.

3. The moduli of absolutely thick representations

In this section, we show that absolute thickness is an open condition in the representation variety. (For representation varieties, see [3])

Let $\mathrm{Rep}_n(G)$ be the representation variety of degree n for a group G over $\mathbb{Z}$. The representation variety represents the following contravariant functor from the category of schemes to the category of sets:

$$\begin{array}{cccl} \mathrm{Rep}_n(G): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \{\text{ a group representation } \rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))\}, \end{array}$$

where $\Gamma(X, \mathcal{O}_X)$ is the ring of global sections on X. The representation variety $\mathrm{Rep}_n(G)$ has the universal n-dimensional representation $\tilde{\rho}$ of G. Let $\mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$ be the Grassmann scheme over $\mathbb{Z}$ representing the contravariant functor

$$\begin{array}{cccl} \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}}): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ W \,\middle|\, W \subseteq \mathcal{O}_X^{\oplus n} \text{ is a subbundle of rank } d \right\}. \end{array}$$

Let us define a subfunctor $X(d, n; G)$ of $\mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$ for $0 < d < n$ by

$$\begin{array}{cccl} X(d, n; G): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ (\rho, W) \,\middle|\, \begin{array}{l} \rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X)), \\ W \subseteq \mathcal{O}_X^{\oplus n} \text{ is a subbundle of rank } d, \\ \text{and } \rho(G)W \subseteq W \end{array} \right\}. \end{array}$$

We show that $X(d, n; G)$ is a closed subscheme of $\mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$.

Lemma 3.1. *For $d = 1$, $X(d, n; G)$ is a closed subscheme of* $\mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$.

Proof. The Grassmann scheme $\mathrm{Gr}(1, \mathbb{A}^n_{\mathbb{Z}})$ can be regarded as $\mathbb{P}_*(\mathbb{A}^n_{\mathbb{Z}}) := \{[w] \mid w$ is a non-zero "vector" of $\mathbb{A}^n_{\mathbb{Z}}\}$. Then

$$\begin{aligned} X(1, n; G) &= \{(\rho, [w]) \mid w \text{ is a non-zero } \rho(G)\text{-eigenvector } \} \\ &= \cap_{g \in G} \{(\rho, [w]) \mid w \text{ is a non-zero } \rho(g)\text{-eigenvector } \}. \end{aligned}$$

The condition that $w \in \mathbb{A}^n$ is a $\rho(g)$-eigenvector can be written by the equations that all 2-minors of the $n \times 2$ matrix $(\rho(g)w, w)$ are 0. Hence the subfunctor $X(1, n; G)$ is a closed subscheme of $\mathrm{Rep}_n(G) \times \mathrm{Gr}(1, \mathbb{A}^n_{\mathbb{Z}})$. □

Proposition 3.2. *For $0 < d < n$, $X(d, n; G)$ is a closed subscheme of* $\mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$.

Proof. The statement is true for $d = 1$ by Lemma 3.1. For $0 < d < n$, by taking the exterior, we get the morphism

$$\begin{array}{rccc} \Phi : & \mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}}) & \to & \mathrm{Rep}_{\binom{n}{d}}(G) \times \mathrm{Gr}(1, \wedge^d \mathbb{A}^n_{\mathbb{Z}}) \\ & (\rho, W) & \mapsto & (\wedge^d \rho, \wedge^d W). \end{array}$$

The subfunctor $X(d, n; G)$ can be obtained by taking the pull-back of the closed subscheme $X(1, \binom{n}{d}; G)$ of $\mathrm{Rep}_{\binom{n}{d}}(G) \times \mathrm{Gr}(1, \wedge^d \mathbb{A}^n_{\mathbb{Z}})$ by Φ. Hence $X(d, n; G)$ is a closed subscheme of $\mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$. □

Let $0 < m < n$. The universal representation $\tilde{\rho}$ on $\mathrm{Rep}_n(G)$ induces an $\binom{n}{m}$-dimensional representation $\wedge^m \tilde{\rho}$ on $\mathrm{Rep}_{\binom{n}{m}}(G)$. This correspondence gives us the canonical morphism $\wedge^m : \mathrm{Rep}_n(G) \to \mathrm{Rep}_{\binom{n}{m}}(G)$ by $\rho \mapsto \wedge^m \rho$. For $0 < d < \binom{n}{m}$, we define the subfunctor $Y(d, \wedge^m(n); G)$ of $\mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$ by

$$\begin{array}{rccc} Y(d, \wedge^m(n); G) : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ (\rho, W) \left| \begin{array}{l} W \subseteq \wedge^m \mathcal{O}_X^{\oplus n} \text{ is a } (\wedge^m \rho)(G)\text{-invariant} \\ \text{subbundle of rank } d \end{array} \right. \right\}. \end{array}$$

Let us define $\phi := \wedge^m \times id : \mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}}) \to \mathrm{Rep}_{\binom{n}{m}}(G) \times \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$ by $(\rho, W) \mapsto (\wedge^m \rho, W)$. The subfunctor $Y(d, \wedge^m(n); G)$ is obtained by taking the pull-back of the closed subscheme $X(d, \binom{n}{m}; G)$ of $\mathrm{Rep}_{\binom{n}{m}}(G) \times \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$ by ϕ. Hence the subfunctor $Y(d, \wedge^m(n); G)$ is a closed subscheme of $\mathrm{Rep}_n(G) \times \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$.

We define the subfunctor $Y(d,\wedge^m(n),\wedge^{n-m}(n);G)$ of $\mathrm{Rep}_n(G)\times \mathrm{Gr}(d,\wedge^m\mathbb{A}^n_{\mathbb{Z}})\times \mathrm{Gr}(\binom{n}{m}-d,\wedge^{n-m}\mathbb{A}^n_{\mathbb{Z}})$ by

$$\begin{array}{rcl} Y(d,\wedge^m(n),\wedge^{n-m}(n);G) & : & (\mathbf{Sch})^{op}\to(\mathbf{Sets}) \\ X & \mapsto & \left\{ (\rho,W_1,W_2) \left| \begin{array}{l} W_1\subseteq \wedge^m\mathcal{O}_X^{\oplus n} \text{ is a } (\wedge^m\rho)(G)\text{-invariant} \\ \text{ subbundle of rank } d, \text{ and} \\ W_2\subseteq \wedge^{n-m}\mathcal{O}_X^{\oplus n} \text{ is a } (\wedge^{n-m}\rho)(G)\text{-invariant} \\ \text{ subbundle of rank } \binom{n}{m}-d \end{array} \right. \right\}. \end{array}$$

Set $X_{n,m,d}(G):=\mathrm{Rep}_n(G)\times\mathrm{Gr}(d,\wedge^m\mathbb{A}^n_{\mathbb{Z}})\times\mathrm{Gr}(\binom{n}{m}-d,\wedge^{n-m}\mathbb{A}^n_{\mathbb{Z}})$. Let us consider the two projections

$$\begin{array}{rcl} \phi_1 & : & X_{n,m,d}(G)\to\mathrm{Rep}_n(G)\times\mathrm{Gr}(d,\wedge^m\mathbb{A}^n_{\mathbb{Z}}) \\ \phi_2 & : & X_{n,m,d}(G)\to\mathrm{Rep}_n(G)\times\mathrm{Gr}(\binom{n}{m}-d,\wedge^{n-m}\mathbb{A}^n_{\mathbb{Z}}). \end{array}$$

Take the pull-backs $\phi_1^{-1}(Y(d,\wedge^m(n);G))$ and $\phi_2^{-1}(Y(\binom{n}{m}-d,\wedge^{n-m}(n);G))$. The subfunctor $Y(d,\wedge^m(n),\wedge^{n-m}(n);G)$ can be obtained as the intersection of these two pull-backs. Therefore $Y(d,\wedge^m(n),\wedge^{n-m}(n);G)$ is a closed subscheme of $X_{n,m,d}(G)$.

Set $\mathrm{Gr}_{n,m,d}:=\mathrm{Gr}(d,\wedge^m\mathbb{A}^n_{\mathbb{Z}})\times\mathrm{Gr}(\binom{n}{m}-d,\wedge^{n-m}\mathbb{A}^n_{\mathbb{Z}})$. Let us consider the perfect pairing on $\mathrm{Gr}_{n,m,d}$:

$$\langle\ ,\ \rangle:(\wedge^m\mathcal{O}^{\oplus n}_{\mathrm{Gr}_{n,m,d}})\otimes_{\mathcal{O}_{\mathrm{Gr}_{n,m,d}}}(\wedge^{n-m}\mathcal{O}^{\oplus n}_{\mathrm{Gr}_{n,m,d}})\to\wedge^n\mathcal{O}^{\oplus n}_{\mathrm{Gr}_{n,m,d}}\cong\mathcal{O}_{\mathrm{Gr}_{n,m,d}}$$

defined by $\langle x,y\rangle:=x\wedge y$. We define the subfunctor $\mathrm{Gr}^{\perp}_{n,m,d}$ of $\mathrm{Gr}_{n,m,d}$ by

$$\mathrm{Gr}^{\perp}_{n,m,d}:=\{(W_1,W_2)\in\mathrm{Gr}_{n,m,d}\mid W_1^{\perp}=W_2\}.$$

For each point $p=(W_1,W_2)\in\mathrm{Gr}_{n,m,d}$, choose a neighbourhood U of p and sections $\{e_i\},\{f_j\}$ on U such that $\langle e_1,e_2,\ldots,e_d\rangle$ is the universal subbundle of $\wedge^m\mathcal{O}^{\oplus n}_{\mathrm{Gr}_{n,m,d}}$ of rank d on U and $W_2=\langle f_1,f_2,\ldots,f_{\binom{n}{m}-d}\rangle$ is the universal subbundle of $\wedge^{n-m}\mathcal{O}^{\oplus n}_{\mathrm{Gr}_{n,m,d}}$ of rank $\binom{n}{m}-d$ on U. The equations $\langle e_i,f_j\rangle=0$ define a closed subscheme structure on $\mathrm{Gr}^{\perp}_{n,m,d}$. Hence $\mathrm{Gr}^{\perp}_{n,m,d}$ is a closed subscheme of $\mathrm{Gr}_{n,m,d}$.

Let us denote by $\phi_3:X_{n,m,d}(G)\to\mathrm{Gr}_{n,m,d}$ the canonical projection. Taking the intersection of $Y(d,\wedge^m(n),\wedge^{n-m}(n);G)$ with the pull-back $\phi_3^{-1}(\mathrm{Gr}^{\perp}_{n,m,d})$, we obtain a closed subscheme $Y(d,\wedge^m(n),\wedge^{n-m}(n);G)^{\perp}$ of $Y(d,\wedge^m(n),\wedge^{n-m}(n);G)$. The closed subscheme $Y(d,\wedge^m(n),\wedge^{n-m}(n);G)^{\perp}$ represents the contravariant functor

$$\begin{array}{rcl} (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \left\{(\rho,W_1,W_2)\in Y(d,\wedge^m(n),\wedge^{n-m}(n);G)(X)\,\middle|\, W_1^{\perp}=W_2\ \right\}. \end{array}$$

For proving openness of absolute m-thickness, we show that realizable subspaces form a closed subset in the Grassmann scheme. We set

$$\mathrm{Gr}(d,\wedge^m\mathbb{A}^n_{\mathbb{Z}})_{real}:=\{W\in\mathrm{Gr}(d,\wedge^m\mathbb{A}^n_{\mathbb{Z}})\mid W\text{ is realizable }\}.$$

More precisely, for a point $x \in \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$, $x \in \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$ if and only if there exists an extension field K of the residue field $k(x)$ of x such that the d-dimensional subspace $W \subseteq \wedge^m K^n$ associated to x is realizable over K.

Proposition 3.3. *The subset* $\mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$ *can be regarded as a closed subscheme of* $\mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$.

Proof. Let us consider the closed subscheme

$$\mathrm{Flag}(1, d, \wedge^m \mathbb{A}^n_{\mathbb{Z}}) := \{([v], W) \in \mathbb{P}_*(\wedge^m \mathbb{A}^n_{\mathbb{Z}}) \times \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}}) \mid v \in W\}$$

of $\mathbb{P}_*(\wedge^m \mathbb{A}^n_{\mathbb{Z}}) \times \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}}) = \mathrm{Gr}(1, \wedge^m \mathbb{A}^n_{\mathbb{Z}}) \times \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$. The scheme $\mathbb{P}_*(\wedge^m \mathbb{A}^n_{\mathbb{Z}})$ has a closed subscheme $\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}})$. Then we obtain the pull-back $p_1^{-1}(\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}}))$ of $\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}})$ by the first projection $p_1 : \mathrm{Flag}(1, d, \wedge^m \mathbb{A}^n_{\mathbb{Z}}) \to \mathbb{P}_*(\wedge^m \mathbb{A}^n_{\mathbb{Z}})$. The subset $\mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$ is the image $p_2(p_1^{-1}(\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}})))$ of the closed subscheme $p_1^{-1}(\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}}))$ by the second projection $p_2 : \mathrm{Flag}(1, d, \wedge^m \mathbb{A}^n_{\mathbb{Z}}) \to \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$. The projection p_2 is proper, and hence we can define a closed subscheme structure on $\mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$. □

Remark 3.4. In the proof of Proposition 3.3, we also see that

$$\mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real} = p_2(p_1^{-1}(\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}}))). \tag{4}$$

The following proposition gives a characterization of $\mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$.

Proposition 3.5. *Let* $x \in \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$. *Let* $\overline{k(x)}$ *be an algebraic closure of the residue field* $k(x)$ *of* x. *Then* $x \in \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$ *if and only if the corresponding* d*-dimensional subspace* $W \otimes_{k(x)} \overline{k(x)} \subseteq \wedge^m \overline{k(x)}^n$ *to* x *is realizable over* $\overline{k(x)}$.

Proof. Let $x \in \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$. If the corresponding d-dimensional subspace $W \otimes_{k(x)} \overline{k(x)} \subseteq \wedge^m \overline{k(x)}^n$ is realizable over $\overline{k(x)}$, then there exists a $\overline{k(x)}$-rational point of $p_1^{-1}(\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}})) \subseteq \mathrm{Flag}(1, d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$ whose image by p_2 corresponds to $W \otimes_{k(x)} \overline{k(x)}$. Then $x \in p_2(p_1^{-1}(\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}}))) = \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$.

Conversely, suppose that $x \in \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$. Setting $\phi := p_2|_{p_1^{-1}(\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}}))}$, we have the following commutative diagram which is a fibre product:

$$\begin{array}{ccc} p_1^{-1}(\mathrm{Gr}(m, \mathbb{A}^n_{\mathbb{Z}})) & \stackrel{\phi}{\to} & \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real} \\ \uparrow & & \uparrow \\ \phi^{-1}(x) & \to & \mathrm{Spec}\, k(x). \end{array}$$

Since ϕ is of finite type, so is $\phi^{-1}(x) \to \mathrm{Spec}\, k(x)$. Note that $\phi^{-1}(x) \neq \emptyset$ by the definition of $\mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$. Then there exists a $\overline{k(x)}$-rational point of $\phi^{-1}(x)$. This implies that the corresponding d-dimensional subspace $W \otimes_{k(x)} \overline{k(x)} \subseteq \wedge^m \overline{k(x)}^n$ is realizable over $\overline{k(x)}$. □

Let $q_2 : X_{n,m,d}(G) \to \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})$ and $q_3 : X_{n,m,d}(G) \to \mathrm{Gr}(\binom{n}{m} - d, \wedge^{n-m} \mathbb{A}^n_{\mathbb{Z}})$ be the second and the third projections. We denote by $Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real}$ the intersection of $Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}$ with $q_2^{-1}(\mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}) \cap q_3^{-1}(\mathrm{Gr}(\binom{n}{m} - d, \wedge^{n-m}\mathbb{A}^n_{\mathbb{Z}})_{real})$. By Proposition 3.3, $Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real}$ can be regarded as a closed subscheme of $Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}$.

Proposition 3.6. *Let $x = (\rho, W_1, W_2) \in Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}$. Let $\overline{k(x)}$ be an algebraic closure of the residue field $k(x)$ of x. Then $x \in Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real}$ if and only if $W_1 \otimes_{k(x)} \overline{k(x)}$ and $W_2 \otimes_{k(x)} \overline{k(x)}$ are realizable over $\overline{k(x)}$.*

Proof. By the definition, $x \in Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real}$ if and only if $q_2(x) \in \mathrm{Gr}(d, \wedge^m \mathbb{A}^n_{\mathbb{Z}})_{real}$ and $q_3(x) \in \mathrm{Gr}(\binom{n}{m} - d, \wedge^{n-m}\mathbb{A}^n_{\mathbb{Z}})_{real}$. It follows from Proposition 3.5 that this condition is equivalent to that $W_1 \otimes_{k(x)} \overline{k(x)}$ and $W_2 \otimes_{k(x)} \overline{k(x)}$ are realizable over $\overline{k(x)}$. □

Let $q_1 : X_{n,m,d}(G) \to \mathrm{Rep}_n(G)$ be the first projection. Since q_1 is proper and $Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real}$ is a closed subscheme of $X_{n,m,d}(G)$, $\mathrm{Rep}_n(G)$ has a closed subscheme $q_1(Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real})$.

Proposition 3.7. *Let $x \in \mathrm{Rep}_n(G)$. Then $x \in q_1(Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real})$ if and only if there exist G-invariant realizable subspaces $W_1 \subseteq \wedge^m \overline{k(x)}^n$ and $W_2 \subseteq \wedge^{n-m}\overline{k(x)}^n$ with respect to the corresponding representation $\rho_x \otimes_{k(x)} \overline{k(x)} : G \to \mathrm{GL}_n(\overline{k(x)})$ such that $\dim W_1 = d, \dim W_2 = \binom{n}{m} - d$, and $W_1^{\perp} = W_2$.*

Proof. First, we prove the "if" part. Suppose that there exist such W_1 and W_2. Then we have a $\overline{k(x)}$-rational point of $Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real}$ whose image by q_1 corresponds to x. Hence $x \in q_1(Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real})$.

Next, we prove the "only if" part. Let $x \in q_1(Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real})$. Set $\psi = q_1|_{Y(d,\wedge^m(n),\wedge^{n-m}(n);G)^{\perp}_{real}} : Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real} \to \mathrm{Rep}_n(G)$. Since ψ is of finite type, so is $\psi^{-1}(x) \to \mathrm{Spec}\, k(x)$. The fibre $\psi^{-1}(x)$ is not empty, and hence there exist W_1 and W_2 with the desired property by Proposition 3.6. □

We can prove the following theorem on absolute m-thickness by Proposition 3.7.

Theorem 3.8. *Let $\rho : G \to \mathrm{GL}_n(k)$ be an n-dimensional representation of G over a field k. For $0 < m < n$, the following conditions are equivalent:*

1. *ρ is absolutely m-thick, in other words, $\rho \otimes_k \overline{k}$ is m-thick for an algebraic closure $\overline{k}$ of k.*
2. *$\rho \otimes_k K$ is m-thick for some algebraically closed field K over k.*
3. *$\rho \otimes_k K$ is m-thick for any algebraically closed field K over k.*

Proof. It is obvious that (3) ⇒ (1) and that (1) ⇒ (2). Let us show that (2) ⇒ (3). Assume that $\rho \otimes_k K$ is m-thick for some algebraically closed field K over k. Note that $\rho \otimes_k \overline{k}$ is also m-thick by Remark 2.15. Suppose that $\rho \otimes_k K'$ is not m-thick for some algebraically closed field K' over k. Let x be the k-rational point of $\mathrm{Rep}_n(G)$ associated to ρ. By Proposition 2.11, there exists a K'-rational point of $Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real}$ for some d whose image by q_1 corresponds to $\rho \otimes_k K'$. Hence $x \in q_1(Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real})$. Then $\rho \otimes_k \overline{k}$ is not m-thick by Proposition 3.7, which is a contradiction. Hence $\rho \otimes_k K'$ is m-thick for any algebraically closed field K' over k. Therefore we have shown that (2) $\Rightarrow$ (3). □

Now we show the openness of absolute m-thickness.

Theorem 3.9. *Let* $\mathrm{Rep}_n(G)$ *be the representation variety of degree* n *for a group* G *over* $\mathbb{Z}$*. For* $0 < m < n$*, the absolutely* m*-thick representations in* $\mathrm{Rep}_n(G)$ *form an open subscheme of* $\mathrm{Rep}_n(G)$*. In particular, the absolutely thick representations in* $\mathrm{Rep}_n(G)$ *form an open subscheme of* $\mathrm{Rep}_n(G)$*.*

Proof. The absolutely m-thick representations form the complement of

$$\bigcup_{0<d<\binom{n}{m}} q_1(Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real})$$

in $\mathrm{Rep}_n(G)$ by Propositions 2.11 and 3.7. Since $q_1(Y(d, \wedge^m(n), \wedge^{n-m}(n); G)^{\perp}_{real})$ is closed for each d, we can verify the openness of absolute m-thickness. We can also prove the openness of absolute thickness by considering all m. □

Let $\mathrm{Rep}_n(G)_{m\text{-thick}}$ be the open subscheme consisting of absolutely m-thick representations of $\mathrm{Rep}_n(G)$. Let $\mathrm{Rep}_n(G)_{\mathrm{thick}}$ be the open subscheme consisting of absolutely thick representations of $\mathrm{Rep}_n(G)$. The open subschemes $\mathrm{Rep}_n(G)_{m\text{-thick}}$ and $\mathrm{Rep}_n(G)_{\mathrm{thick}}$ are contained in the representation variety of absolutely irreducible representations $\mathrm{Rep}_n(G)_{\mathrm{air}}$. We have group actions of the group scheme PGL_n on these schemes by the conjugation $\rho \mapsto P^{-1}\rho P$. By [3, Theorem 1.3], there exists a universal geometric quotient $\mathrm{Ch}_n(G)_{\mathrm{air}}$ of $\mathrm{Rep}_n(G)_{\mathrm{air}}$ by PGL_n and the quotient morphism $\mathrm{Rep}_n(G)_{\mathrm{air}} \to \mathrm{Ch}_n(G)_{\mathrm{air}}$ is a PGL_n-principal fibre bundle. Hence we have the following theorem:

Theorem 3.10. *For each* $0 < m < n$*, there exists a universal geometric quotient* $\mathrm{Ch}_n(G)_{m\text{-thick}}$ *of* $\mathrm{Rep}_n(G)_{m\text{-thick}}$ *by* PGL_n*. Moreover, there exists a universal geometric quotient* $\mathrm{Ch}_n(G)_{\mathrm{thick}}$ *of* $\mathrm{Rep}_n(G)_{\mathrm{thick}}$ *by* PGL_n*. The quotient morphisms* $\mathrm{Rep}_n(G)_{m\text{-thick}} \to \mathrm{Ch}_n(G)_{m\text{-thick}}$ *and* $\mathrm{Rep}_n(G)_{\mathrm{thick}} \to \mathrm{Ch}_n(G)_{\mathrm{thick}}$ *are* PGL_n*-principal fibre bundles.*

We also have the same results on absolutely dense representations as absolutely thick representations.

Proposition 3.11. *For* $0 < m < n$*, the absolutely* m*-dense representations in* $\mathrm{Rep}_n(G)$ *form an open subscheme of* $\mathrm{Rep}_n(G)$*. In particular, the absolutely dense representations in* $\mathrm{Rep}_n(G)$ *form an open subscheme of* $\mathrm{Rep}_n(G)$*.*

Proof. We define the morphism $\wedge^m : \mathrm{Rep}_n(G) \to \mathrm{Rep}_{\binom{n}{m}}(G)$ by $\rho \mapsto \wedge^m \rho$. The inverse image of the open subscheme $\mathrm{Rep}_{\binom{n}{m}}(G)_{\mathrm{air}}$ by $\wedge^m$ coincides with the absolutely m-dense representations in $\mathrm{Rep}_n(G)$. Hence it is open. Considering all m, we see that the absolutely dense representations in $\mathrm{Rep}_n(G)$ is also open. □

Let $\mathrm{Rep}_n(G)_{m\text{-dense}}$ be the open subscheme consisting of absolutely m-dense representations of $\mathrm{Rep}_n(G)$. Let $\mathrm{Rep}_n(G)_{\mathrm{dense}}$ be the open subscheme consisting of absolutely dense representations of $\mathrm{Rep}_n(G)$. The open subschemes $\mathrm{Rep}_n(G)_{m\text{-dense}}$ and $\mathrm{Rep}_n(G)_{\mathrm{dense}}$ are contained in the representation variety of absolutely irreducible representations $\mathrm{Rep}_n(G)_{\mathrm{air}}$. In the same way as absolutely thick representations, we have the following theorem:

Theorem 3.12. *For each $0 < m < n$, there exists a universal geometric quotient $\mathrm{Ch}_n(G)_{m\text{-dense}}$ of $\mathrm{Rep}_n(G)_{m\text{-dense}}$ by PGL_n. Moreover, there exists a universal geometric quotient $\mathrm{Ch}_n(G)_{\mathrm{dense}}$ of $\mathrm{Rep}_n(G)_{\mathrm{dense}}$ by PGL_n. The quotient morphisms $\mathrm{Rep}_n(G)_{m\text{-dense}} \to \mathrm{Ch}_n(G)_{m\text{-dense}}$ and $\mathrm{Rep}_n(G)_{\mathrm{dense}} \to \mathrm{Ch}_n(G)_{\mathrm{dense}}$ are PGL_n-principal fibre bundles.*

Summarizing the results above, we have the following diagrams:

$$\begin{array}{ccccc} \mathrm{Rep}_n(G)_{m\text{-dense}} & \subseteq & \mathrm{Rep}_n(G)_{m\text{-thick}} & \subseteq & \mathrm{Rep}_n(G)_{\mathrm{air}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ch}_n(G)_{m\text{-dense}} & \subseteq & \mathrm{Ch}_n(G)_{m\text{-thick}} & \subseteq & \mathrm{Ch}_n(G)_{\mathrm{air}} \end{array}$$

and

$$\begin{array}{ccccc} \mathrm{Rep}_n(G)_{\mathrm{dense}} & \subseteq & \mathrm{Rep}_n(G)_{\mathrm{thick}} & \subseteq & \mathrm{Rep}_n(G)_{\mathrm{air}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ch}_n(G)_{\mathrm{dense}} & \subseteq & \mathrm{Ch}_n(G)_{\mathrm{thick}} & \subseteq & \mathrm{Ch}_n(G)_{\mathrm{air}}. \end{array}$$

Remark 3.13. For a representation $\rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$ of a group G on a scheme X, ρ is called *absolutely m-thick* (resp. *absolutely thick*) if the induced representation $\rho \otimes k(x) : G \to \mathrm{GL}_n(k(x))$ is absolutely m-thick (resp. absolutely thick) for each $x \in X$, where $k(x)$ is the residue field of x. Similarly, ρ is called *absolutely m-dense* (resp. *absolutely dense*) if the induced representation $\rho \otimes k(x) : G \to \mathrm{GL}_n(k(x))$ is absolutely m-dense (resp. absolutely dense) for each $x \in X$. The scheme $\mathrm{Rep}_n(G)_{m\text{-thick}}$ (resp. $\mathrm{Rep}_n(G)_{\mathrm{thick}}$, $\mathrm{Rep}_n(G)_{m\text{-dense}}$, $\mathrm{Rep}_n(G)_{\mathrm{dense}}$) represents the contravariant functor from the category of schemes to the category of sets which maps each scheme to the set of n-dimensional absolutely m-thick (resp. absolutely thick, absolutely m-dense, absolutely dense) representations of G on X.

4. Realizable subspaces

In this section, we discuss realizable subspaces in detail. We introduce the r-number $r(\wedge^m(n))$ which is closely related to thickness. In some cases, we can calculate $r(\wedge^m(n))$.

Lemma 4.1. *Let V be an n-dimensional vector space over an algebraically closed field k. Let W be a vector subspace of $\wedge^m V$ with $0 < m < n$. If $\mathrm{codim} W \leq m(n-m)$, then W is realizable, in other words, there exists an m-dimensional vector subspace V_1 of V such that $\wedge^m V_1 \in W$.*

Proof. Remark that $\wedge^m V_1 \in \wedge^m V$ can be defined up to scalar multiplication. The Grassmann variety $\mathrm{Gr}(m, V) \subset \mathbb{P}_*(\wedge^m V)$ has dimension $m(n-m)$. Since the subspace $\mathbb{P}_*(W) \subset \mathbb{P}_*(\wedge^m V)$ has codimension $\leq m(n-m)$, the intersection $\mathbb{P}_*(W) \cap \mathrm{Gr}(m, V)$ is not empty. Hence there exists an m-dimensional subspace V_1 such that $\wedge^m V_1 \in W$. □

Proposition 4.2. *Let V be an n-dimensional vector space over an algebraically closed field k. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. If $\wedge^m V$ has a $(\wedge^m \rho)(G)$-invariant realizable subspace W of $\dim W \leq m(n-m)$, then ρ is not m-thick.*

Proof. Let us consider $W^\perp \subseteq \wedge^{n-m} V$. Since $\dim W \leq m(n-m)$, $\mathrm{codim} W^\perp \leq m(n-m)$. By Lemma 4.1, $W^\perp$ is realizable. Hence ρ is not m-thick because of Proposition 2.11. □

Definition 4.3. For $0 < m < n$, we define the *r-number* $r(\wedge^m(n))$ by

$$r(\wedge^m(n)) := \min \left\{ \dim W \;\middle|\; \begin{array}{l} \text{there exists an } n\text{-dimensional irreducible} \\ \text{representation } \rho : G \to \mathrm{GL}(V) \text{ of a group } G \\ \text{over a field } k \text{ such that } W \text{ is a } G\text{-invariant} \\ \text{realizable subspace of } \wedge^m V \end{array} \right\}.$$

For convenience, we set $r(\wedge^0(n)) = 1$ and $r(\wedge^n(n)) = 1$ for each positive integer n.

For a real number x, we denote by $[x]$ the largest integer which is equal to or less than x.

Proposition 4.4. *For $0 < m < n$, $r(\wedge^m(n)) \geq [\frac{n-1}{m}] + 1$.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional irreducible representation of a group G. Let $W \subseteq \wedge^m V$ be a G-invariant realizable subspace. We show that $\dim W \geq [\frac{n-1}{m}] + 1$. Since W is realizable, there exists a basis $e_1, e_2, \ldots, e_n$ of V such that $x := e_1 \wedge e_2 \wedge \cdots \wedge e_m \in W$. We define $g_i \in G$ for $1 \leq i \leq [\frac{n-1}{m}] + 1$ in the following way: Let $g_1 := e \in G$. If $g_i \in G$ is determined for $i \leq k$, choose $g_{k+1} \in G$ such that $\rho(g_{k+1})e_1$ is not contained in the subspace V_k spanned by $\{\rho(g_i)e_j \mid 1 \leq i \leq k, 1 \leq j \leq m\}$ of V. This procedure is possible, since $\dim V_k \leq km \leq [\frac{n-1}{m}]m < n$ and the set $\{\rho(g)e_1 \mid g \in G\}$ spans V because of the irreducibility of ρ. In this way, $g_1, g_2, \ldots, g_{[\frac{n-1}{m}]+1}$ can be chosen.

We claim that $(\wedge^m\rho)(g_1)x, (\wedge^m\rho)(g_2)x, \dots, (\wedge^m\rho)(g_{[\frac{n-1}{m}]+1})x \in W$ are linearly independent. Let $\sum a_i(\wedge^m\rho)(g_i)x = 0$ for $a_i \in k$. By using

$$(\wedge^m\rho)(g_i)x \wedge \rho(g_{i+1})e_1 \wedge \rho(g_{i+2})e_1 \wedge \cdots \wedge \rho(g_{[\frac{n-1}{m}]+1})e_1 \neq 0 \text{ and}$$
$$(\wedge^m\rho)(g_i)x \wedge \rho(g_i)e_1 \wedge \rho(g_{i+1})e_1 \wedge \rho(g_{i+2})e_1 \wedge \cdots \wedge \rho(g_{[\frac{n-1}{m}]+1})e_1 = 0,$$

we see that $a_i = 0$ for each i, which implies the claim. Hence $\dim W \geq [\frac{n-1}{m}]+1$. □

Corollary 4.5. *If $0 < m < n$, then $r(\wedge^m(n)) \geq 2$. In particular, if ρ is an n-dimensional irreducible representation, then $\wedge^m\rho$ has no 1-dimensional G-invariant realizable subspace.*

Proof. The statement follows from that $r(\wedge^m(n)) \geq [\frac{n-1}{m}] + 1 \geq 2$. □

If m divides n, then we can prove that $r(\wedge^m(n)) = \frac{n}{m}$. For proving this, we need to make some preparations.

Lemma 4.6. *Let $f : V \to V$ be a linear endomorphism on an n-dimensional vector space V over a field k. Suppose that f has n distinct eigenvalues $\alpha_1, \dots, \alpha_n \in k$. Let $e_1, \dots, e_n \in V$ be eigenvectors associated to $\alpha_1, \dots, \alpha_n$, respectively. Then for any f-invariant subspace W of V, there exists a subset I of $\{1, 2, \dots, n\}$ such that $W = \oplus_{i\in I} k \cdot e_i$.*

Proof. For an f-invariant subspace W, we define a subset I of $\{1, 2, \dots, n\}$ by $I := \{i \mid \text{there exists } \sum_{j=1}^n a_j e_j \in W \text{ such that } a_i \neq 0\}$. It is clear that $W \subseteq \oplus_{i\in I} k \cdot e_i$. We show that $W \supseteq \oplus_{i\in I} k \cdot e_i$. For each $i \in I$, there exists a vector $x = \sum_{j=1}^n a_j e_j \in W$ such that $a_i \neq 0$. Set $J := \{j \mid a_j \neq 0\} = \{j_1, j_2, \dots, j_m\}$ and $m := \sharp J$. Note that $i \in J$. Since $f(x) = \sum_{j\in J} \alpha_j a_j e_j$, $f^2(x) = \sum_{j\in J} \alpha_j^2 a_j e_j, \dots, f^{m-1}(x) = \sum_{j\in J} \alpha_j^{m-1} a_j e_j$, we have

$$\begin{pmatrix} x \\ f(x) \\ f^2(x) \\ \vdots \\ f^{m-1}(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_{j_1} & \alpha_{j_2} & \cdots & \alpha_{j_m} \\ \alpha_{j_1}^2 & \alpha_{j_2}^2 & \cdots & \alpha_{j_m}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{j_1}^{m-1} & \alpha_{j_2}^{m-1} & \cdots & \alpha_{j_m}^{m-1} \end{pmatrix} \begin{pmatrix} a_{j_1} e_{j_1} \\ a_{j_2} e_{j_2} \\ a_{j_3} e_{j_3} \\ \vdots \\ a_{j_m} e_{j_m} \end{pmatrix}.$$

The matrix $(\alpha_{j_t}^{s-1})_{1\leq s,t\leq m}$ is invertible, and hence the vector $a_{j_s} e_{j_s}$ can be written as a linear combination of $x, f(x), f^2(x), \dots, f^{m-1}(x)$ for each $1 \leq s \leq m$. In particular, $e_i \in W$. This implies that $W \supseteq \oplus_{i\in I} k \cdot e_i$. So we have proved the lemma. □

Lemma 4.7. *Let V be a vector space over an infinite field k. For any non-zero vector $v \in V$ and a finite subset $S \subset k^\times$, there exists $f \in \mathrm{GL}(V)$ satisfying the following conditions:*

(1) *There exists a basis $\{v_1, v_2, \dots, v_n\}$ of V such that v_i is an eigenvector of f with eigenvalues $\beta_i \in k^\times \setminus S$ for $1 \leq i \leq n$.*

(2) $\beta_1, \beta_2, \dots, \beta_n$ *are distinct.*
(3) $v = v_1 + v_2 + \cdots + v_n$.

In particular, v is not contained in any proper f-invariant subspaces.

Proof. Let us take vectors $v_1, v_2, \dots, v_{n-1} \in V$ such that $\{v, v_1, v_2, \dots, v_{n-1}\}$ is a basis of V. Put $v_n := v - v_1 - v_2 - \cdots - v_{n-1}$. Then $\{v_1, v_2, \dots, v_n\}$ is a basis of V and $v = v_1 + v_2 + \cdots + v_n$. Let us choose distinct elements $\beta_1, \beta_2, \dots, \beta_n \in k^\times \setminus S$. We define $f \in \mathrm{GL}(V)$ by $f(v_i) = \beta_i v_i$ for $1 \le i \le n$. By Lemma 4.6, for any proper f-invariant subspace W, there exists a proper subset I of $\{1, 2, \dots, n\}$ such that $W = \oplus_{i \in I} k \cdot v_i$. Hence $v = v_1 + v_2 + \cdots + v_n$ is not contained in W. This completes the proof. □

Lemma 4.8. *Let k be a field. Let $A_1, A_2, \dots, A_\ell \in \mathrm{GL}_m(k)$. Set $C := A_\ell A_{\ell-1} \cdots A_2 A_1$ and*

$$X = \begin{pmatrix} 0_m & 0_m & 0_m & \cdots & 0_m & A_\ell \\ A_1 & 0_m & 0_m & \cdots & 0_m & 0_m \\ 0_m & A_2 & 0_m & \cdots & 0_m & 0_m \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \cdots & A_{\ell-1} & 0_m \end{pmatrix} \in \mathrm{GL}_n(k),$$

where $n = \ell m$ with $\ell \ge 2$. Suppose that the eigenvalues $\alpha_1, \dots, \alpha_m$ of C are distinct and that $\sharp\{z \in k \mid z^\ell = \alpha_i\} = \ell$ for each $1 \le i \le m$. Then for each ℓ-th root $\xi_{i,j}$ of α_i ($1 \le i \le m, 1 \le j \le \ell$) and for each eigenvector v_i of C with respect to α_i, the vector

$$w_{i,j} := {}^t(\xi_{i,j}^{\ell-1} v_i,\ \xi_{i,j}^{\ell-2} A_1 v_i,\ \xi_{i,j}^{\ell-3} (A_2 A_1) v_i, \dots,\ \xi_{i,j} (A_{\ell-2} \cdots A_2 A_1) v_i, (A_{\ell-1} A_{\ell-2} \cdots A_2 A_1) v_i)$$

is an eigenvector of X with respect to the eigenvalue $\xi_{i,j}$. Conversely, all eigenvectors of X can be obtained in this way (up to scalar multiplication).

Proof. It is easy to check that $X w_{i,j} = \xi_{i,j} w_{i,j}$. The statement follows from that $\{\xi_{i,j} \mid 1 \le i \le m, 1 \le j \le \ell\}$ forms the set of n distinct eigenvalues of X. □

Let $\mathrm{F}_1 = \langle \alpha \rangle$ be the free group of rank 1. By Proposition 3.2, $X(d, n; \mathrm{F}_1)$ is a closed subscheme of $\mathrm{Rep}_n(\mathrm{F}_1) \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$. Here recall that $X(d, n; \mathrm{F}_1) = \{(\rho, W) \mid W$ is a d-dimensional $\rho(G)$-invariant subbundle of $\mathbb{A}^n\}$. Let $U(d, n) := U(d, n; \mathrm{F}_1)$ be the complement of $X(d, n; \mathrm{F}_1)$ in $\mathrm{Rep}_n(\mathrm{F}_1) \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$. Note that $\mathrm{Rep}_n(\mathrm{F}_1) = \mathrm{GL}_n$ and that $U(d, n) = \{(A, W) \mid W$ is not A-invariant$\} \subseteq \mathrm{GL}_n \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$. For a X-valued point ϕ of $\mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$ with a scheme X, denote by $\phi^*(W) \subset \mathcal{O}_X^{\oplus n}$ the subbundle of rank d induced by ϕ on X. Then we have the following diagram

$$\begin{array}{ccc} \mathrm{GL}_{n,\phi} := \{(A,x) \in \mathrm{GL}_n \times X \mid \phi^*(W)_x \text{ is not } A\text{-invariant}\} & \to & U(d,n) \\ \downarrow & & \downarrow \\ \mathrm{GL}_n \times X & \stackrel{id\times\phi}{\to} & \mathrm{GL}_n \times \mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}}), \end{array}$$

which is a fibre product. Hence $\mathrm{GL}_{n,\phi}$ is an open subscheme of $\mathrm{GL}_n \times X$.

In particular, for a geometric point W of $\mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$, we have:

Proposition 4.9. *Let k be an algebraically closed field. Let W be a k-rational point of $\mathrm{Gr}(d, \mathbb{A}^n_{\mathbb{Z}})$. Then the subset $\{A \in \mathrm{GL}_n(k) \mid W \text{ is not } A\text{-invariant}\}$ is an open subscheme of $\mathrm{GL}_n(k)$.*

Proposition 4.10. *Let ℓ and m be positive integers with $\ell, m \geq 2$. Set $n = \ell m$. Let k be an algebraically closed field such that $\mathrm{ch}\, k$ does not divide ℓ. Then there exists an irreducible representation $\rho : \mathrm{F}_2 \to \mathrm{GL}_n(k)$ of the free group F_2 of rank 2 such that $\wedge^m \rho$ has a realizable invariant subspace of dimension ℓ. Moreover, there exists an irreducible representation $\rho : \mathrm{F}_2 \to \mathrm{GL}_n(k)$ such that ρ is neither m-thick nor ℓ-thick.*

Proof. Let $\mathrm{F}_2 = \langle \alpha, \beta \rangle$. For constructing ρ, we need to determine $A := \rho(\alpha), B := \rho(\beta) \in \mathrm{GL}_n(k)$. The group $\mathrm{GL}_n(k)$ acts canonically on k^n. Let $e_1, e_2, \ldots, e_n$ be the canonical basis of k^n. Set

$$A = \begin{pmatrix} 0_m & 0_m & 0_m & \cdots & 0_m & A' \\ I_m & 0_m & 0_m & \cdots & 0_m & 0_m \\ 0_m & I_m & 0_m & \cdots & 0_m & 0_m \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \cdots & I_m & 0_m \end{pmatrix}, B = \begin{pmatrix} 0_m & 0_m & 0_m & \cdots & 0_m & B_\ell \\ B_1 & 0_m & 0_m & \cdots & 0_m & 0_m \\ 0_m & B_2 & 0_m & \cdots & 0_m & 0_m \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \cdots & B_{\ell-1} & 0_m \end{pmatrix},$$

where $A', B_1, \ldots, B_\ell \in \mathrm{GL}_m(k)$ will be suitably chosen. Let us define $\Phi : \mathrm{GL}_m(k) \times \cdots \times \mathrm{GL}_m(k) = \mathrm{GL}_m(k)^\ell \to \mathrm{GL}_n(k)$ by $(B_1, B_2, \ldots, B_\ell) \mapsto B$.

First, we show that ρ is not m-thick. Let $W := \langle e_1 \wedge e_2 \wedge \cdots \wedge e_m, e_{m+1} \wedge \cdots \wedge e_{2m}, e_{2m+1} \wedge \cdots \wedge e_{3m}, \ldots, e_{(\ell-1)m+1} \wedge \cdots \wedge e_n \rangle \subseteq \wedge^m V$. Note that $\wedge^m \rho$ has a realizable invariant subspace W of dimension ℓ. Since $\ell \leq m(n-m)$, ρ is not m-thick by Proposition 4.2.

Second, we show that ρ is not ℓ-thick. For $1 \leq i \leq \ell$, we put $J_i := \{(i-1)m+1, (i-1)m+2, \ldots, im\}$. Then $J_1 \sqcup J_2 \sqcup \cdots \sqcup J_\ell = \{1, 2, \ldots, n\}$. Let Y be the subspace of $\wedge^\ell V$ generated by $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell} \mid i_1 \in J_1, i_2 \in J_2, \ldots, i_\ell \in J_\ell\}$. Note that Y is an m^ℓ-dimensional $(\wedge^\ell \rho)(\mathrm{F}_2)$-invariant realizable subspace of $\wedge^\ell V$. The subspace $Y^\perp$ of $\wedge^{n-\ell} V$ contains $e_1 \wedge e_2 \wedge \cdots \wedge e_m \wedge v'$ for any $v' \in \wedge^{n-\ell-m} V$. In particular, $Y^\perp$ is realizable. By Proposition 2.11, ρ is not ℓ-thick.

Finally, we show that ρ is irreducible if $A', B_1, \ldots, B_\ell$ are suitably chosen. Let $A' = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m)$, where $\alpha_1, \ldots, \alpha_m \in k^\times$ are distinct. For each ℓ-th root

$\xi_{i,j}$ of α_i $(1 \leq i \leq m, 1 \leq j \leq \ell)$, we define $w_{i,j} := {}^t(\xi_{i,j}^{\ell-1}e'_i, \xi_{i,j}^{\ell-2}e'_i, \ldots, e'_i)$ as in Lemma 4.8, where $A_1 = A_2 = \cdots A_{\ell-1} = I_m$ and $A_\ell = A'$. Here we use $e'_1, \ldots, e'_m$ as the canonical basis of k^m in the sequel. Then $w_{i,j}$ is an eigenvector of A.

By Lemma 4.6, for any A-invariant subspace W of k^n, there exists a subset I of $\{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq \ell\}$ such that $W = W_I := \oplus_{(i,j)\in I} k \cdot w_{i,j}$. For proving ρ is irreducible, it suffices to show that B does not keep any non-trivial A-invariant subspace W_I invariant. For each non-trivial A-invariant subspace W_I, we set $\mathrm{GL}_n(k)_I := \{B \in \mathrm{GL}_n(k) \mid W_I \text{ is not } B\text{-invariant }\}$. By Proposition 4.9, $\mathrm{GL}_n(k)_I$ is an open subscheme of $\mathrm{GL}_n(k)$. Let us prove the claim that the open subset $\Phi^{-1}(\mathrm{GL}_n(k)_I) \subseteq \mathrm{GL}_m(k)^\ell$ is not empty for each non-empty proper subset I of $\{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq \ell\}$. If $\Phi^{-1}(\mathrm{GL}_n(k)_I) \neq \emptyset$ for each I, then $\cap_I \Phi^{-1}(\mathrm{GL}_n(k)_I) \neq \emptyset$ because $\mathrm{GL}_m(k)^\ell$ is irreducible. Then by taking $(B_1, \ldots, B_\ell) \in \cap_I \Phi^{-1}(\mathrm{GL}_n(k)_I)$, we obtain an irreducible representation ρ, which completes the proof.

For proving the claim that $\Phi^{-1}(\mathrm{GL}_n(k)_I) \neq \emptyset$, take some $(i_0, j_0) \in I$. By Lemma 4.7, there exist a basis $\{v_1, v_2, \ldots, v_m\}$ of k^m and $f : k^m \to k^m$ such that $f(v_i) = \beta_i v_i$ $(1 \leq i \leq m)$ and $e'_{i_0} = v_1 + \cdots + v_m$. Here $\beta_1, \ldots, \beta_m$ are distinct elements in $k^\times \setminus \{\alpha_i \mid 1 \leq i \leq m\}$. Let $B_\ell \in \mathrm{GL}_m(k)$ be the corresponding matrix to f. Set $B_1 = B_2 = \cdots = B_{\ell-1} = I_m$ and $B = \Phi(B_1, \ldots, B_\ell)$. For each ℓ-th root η_{ij} of β_i $(1 \leq i \leq m, 1 \leq j \leq \ell)$, put $w'_{i,j} := {}^t(\eta_{i,j}^{\ell-1}v_i, \eta_{i,j}^{\ell-2}v_i, \ldots, v_i) \in k^n$. By Lemma 4.8, $Bw'_{i,j} = \eta_{i,j}w'_{i,j}$ for each i,j. Since $\{w'_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq \ell\}$ is a basis of k^n, we can write $w_{i_0,j_0} = \sum c_{i,j}w'_{i,j}$ for $c_{i,j} \in k$. If $c_{i,j} \neq 0$ for all i,j, then $w_{i_0,j_0} \in W_I$ is not contained in any non-trivial B-invariant subspaces by Lemma 4.6. In particular, W_I is not B-invariant and $(B_1, \ldots, B_\ell) \in \Phi^{-1}(\mathrm{GL}_n(k)_I)$, which implies the claim. Hence we only need to show that $c_{i,j} \neq 0$ for all i,j.

Let us show that $c_{i,j} \neq 0$. For each $1 \leq i \leq m$, we define the ℓ-dimensional subspace $U_i := \langle {}^t(v_i, 0, 0, \ldots, 0), {}^t(0, v_i, 0, \ldots, 0), \ldots, {}^t(0, 0, 0, \ldots, v_i)\rangle \subset k^n$. Let $p_i : k^n = U_1 \oplus \cdots \oplus U_m \to U_i$ be the projection onto U_i. Since $U_i = \oplus_{1\leq j\leq \ell}\, k \cdot w'_{i,j}$,

$$p_i(w_{i_0,j_0}) = p_i({}^t(\xi_{i_0,j_0}^{\ell-1}e'_{i_0}, \xi_{i_0,j_0}^{\ell-2}e'_{i_0}, \ldots, e'_{i_0})) = \sum_{1\leq j\leq \ell} c_{i,j}w'_{i,j}.$$

On the other hand, $e'_{i_0} = v_1 + \cdots + v_m$ and hence

$${}^t(\xi_{i_0,j_0}^{\ell-1}v_i, \xi_{i_0,j_0}^{\ell-2}v_i, \ldots, v_i) = \sum_{1\leq j\leq \ell} c_{i,j}w'_{i,j}.$$

Then we have

$$\begin{pmatrix} \eta_{i,1}^{\ell-1} & \eta_{i,2}^{\ell-1} & \cdots & \eta_{i,\ell}^{\ell-1} \\ \eta_{i,1}^{\ell-2} & \eta_{i,2}^{\ell-2} & \cdots & \eta_{i,\ell}^{\ell-2} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c_{i,1} \\ c_{i,2} \\ \vdots \\ c_{i,\ell} \end{pmatrix} = \begin{pmatrix} \xi_{i_0,j_0}^{\ell-1} \\ \xi_{i_0,j_0}^{\ell-2} \\ \vdots \\ 1 \end{pmatrix}.$$

By Cramer's rule,

$$c_{i,j} = \det \begin{pmatrix} \eta_{i,1}^{\ell-1} & \eta_{i,2}^{\ell-1} & \cdots & \overset{\overset{j}{\smile}}{\xi_{i_0,j_0}^{\ell-1}} & \cdots & \eta_{i,\ell}^{\ell-1} \\ \eta_{i,1}^{\ell-2} & \eta_{i,2}^{\ell-2} & \cdots & \xi_{i_0,j_0}^{\ell-2} & \cdots & \eta_{i,\ell}^{\ell-2} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & 1 \end{pmatrix} \cdot \det \begin{pmatrix} \eta_{i,1}^{\ell-1} & \eta_{i,2}^{\ell-1} & \cdots & \eta_{i,\ell}^{\ell-1} \\ \eta_{i,1}^{\ell-2} & \eta_{i,2}^{\ell-2} & \cdots & \eta_{i,\ell}^{\ell-2} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}^{-1}.$$

The Vandermonde determinant is not 0 because $\eta_{i,j}$ and ξ_{i_0,j_0} are distinct. Hence $c_{i,j} \neq 0$. Therefore we have completed the proof. □

Corollary 4.11. *If m divides n, then $r(\wedge^m(n)) = [\frac{n-1}{m}] + 1 = \frac{n}{m}$.*

Proof. If $m = 1$ or $n = m$, then the statement is trivial. Let $n > m \geq 2$. By Proposition 4.10, there exists an n-dimensional irreducible representation ρ of F_2 such that $\wedge^m \rho$ has a realizable invariant subspace of dimension n/m. Hence we have $r(\wedge^m(n)) = n/m$ by Proposition 4.4. □

The following remark was suggested by the referee.

Remark 4.12. Proposition 4.10 shows that there exists an irreducible representation $\rho : \mathrm{F}_2 \to \mathrm{GL}_n(k)$ such that ρ is neither m-thick nor ℓ-thick, where $n = \ell m$ for $\ell, m \geq 2$. However, we can easily construct an irreducible representation $\rho : \mathrm{F}_3 \to \mathrm{GL}_n(k)$ of the free group $\mathrm{F}_3 = \langle \alpha, \beta, \gamma \rangle$ of rank 3 such that ρ is neither m-thick nor ℓ-thick. Indeed, set $\rho(\alpha) = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$,

$$\rho(\beta) = \begin{pmatrix} B' & 0_{m,n-m} \\ 0_{n-m,m} & I_{n-m} \end{pmatrix}, \ \rho(\gamma) = \begin{pmatrix} 0_m & 0_m & 0_m & \cdots & 0_m & I_m \\ I_m & 0_m & 0_m & \cdots & 0_m & 0_m \\ 0_m & I_m & 0_m & \cdots & 0_m & 0_m \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \cdots & I_m & 0_m \end{pmatrix},$$

where $\alpha_1, \dots, \alpha_n \in k^\times$ are distinct and $B' = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathrm{GL}_m(k)$.

We can show that $\rho : \mathrm{F}_3 \to \mathrm{GL}_n(k)$ is irreducible by Lemma 4.6. We can also prove that ρ is neither m-thick nor ℓ-thick, and that $W := \langle e_1 \wedge e_2 \wedge \cdots \wedge e_m, e_{m+1} \wedge \cdots \wedge e_{2m}, e_{2m+1} \wedge \cdots \wedge e_{3m}, \dots, e_{(\ell-1)m+1} \wedge \cdots \wedge e_n \rangle \subseteq \wedge^m V$ is a realizable $(\wedge^m \rho)(\mathrm{F}_3)$-invariant subspace of dimension ℓ as in the proof of Proposition 4.10.

By the definition, it is obvious that $r(\wedge^m(n)) \leq \binom{n}{m}$. The following proposition gives us a non-trivial upper bound of $r(\wedge^m(n))$.

Proposition 4.13. *For $0 < m < n$, $r(\wedge^m(n)) \leq n$.*

Proof. Let a and b are distinct non-zero elements of a field k. Assume that $\sharp\{c \in k \mid c^n = a\} = \sharp\{c \in k \mid c^n = b\} = n$. Let $\{\xi_i \mid 1 \leq i \leq n\}$ and $\{\eta_i \mid 1 \leq i \leq n\}$ be the n-th roots of a and b, respectively. We define an n-dimensional representation ρ of the free group $\mathrm{F}_2 = \langle \alpha, \beta \rangle$ by

$$\rho(\alpha) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & b \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Set $v_i = {}^t(\xi_i^{n-1}, \xi_i^{n-2}, \ldots, \xi_i, 1)$ and $w_i = {}^t(\eta_i^{n-1}, \eta_i^{n-2}, \ldots, \eta_i, 1)$. It is easy to check that $\rho(\alpha)v_i = \xi_i v_i$ and $\rho(\beta)w_i = \eta_i w_i$ for $1 \leq i \leq n$ and that $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are bases of k^n. In a similar way as the last part of the proof of Proposition 4.10, we can prove that $\rho : \mathrm{F}_2 \to \mathrm{GL}_n(k)$ is irreducible. Indeed, let V be a non-zero subspace of k^n which is invariant under $\rho(\alpha)$ and $\rho(\beta)$. By Lemma 4.6, there exist subsets I and J of $\{1, 2, \ldots, n\}$ such that $V = \oplus_{i \in I} k \cdot v_i = \oplus_{j \in J} k \cdot w_j$. Suppose that $w_j \in V$. Put $w_j = \sum c_i v_i$. Then we have

$$\begin{pmatrix} \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \\ \xi_1^{n-2} & \xi_2^{n-2} & \cdots & \xi_n^{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \eta_j^{n-1} \\ \eta_j^{n-2} \\ \vdots \\ 1 \end{pmatrix}.$$

By Cramer's rule,

$$c_i = \det \begin{pmatrix} \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \overset{\overset{i}{\vee}}{\eta_j^{n-1}} & \cdots & \xi_n^{n-1} \\ \xi_1^{n-2} & \xi_2^{n-2} & \cdots & \eta_j^{n-2} & \cdots & \xi_n^{n-2} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & 1 \end{pmatrix} \cdot \det \begin{pmatrix} \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \\ \xi_1^{n-2} & \xi_2^{n-2} & \cdots & \xi_n^{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}^{-1}.$$

The Vandermonde determinant is not 0 because ξ_i and η_j are distinct, and hence $c_i \neq 0$ for $1 \leq i \leq n$. We see that $I = \{1, 2, \ldots, n\}$ and that $V = k^n$. This implies that ρ is irreducible.

Let $e_1, e_2, \ldots, e_n$ be the canonical basis of k^n. For $0 < m < n$, define an n-dimensional subspace W_m of $\wedge^m k^n$ by

$$W_m := \langle e_1 \wedge e_2 \wedge \cdots \wedge e_m, e_2 \wedge e_3 \wedge \cdots \wedge e_{m+1}, \ldots, e_i \wedge e_{i+1} \wedge \cdots \wedge e_{i+m-1}, \ldots, e_{n-1} \wedge e_n \wedge e_1 \wedge \cdots \wedge e_{m-2}, e_n \wedge e_1 \wedge \cdots \wedge e_{m-1} \rangle \subset \wedge^m k^n.$$

Then W_m is an F_2-invariant realizable subspace of $\wedge^m k^n$. Hence $r(\wedge^m(n)) \le \dim W_m = n$. This completes the proof. □

We prepare some basic results on perfect pairings for determining some $r(\wedge^m(n))$. In the sequel, by a G-module we understand a finite-dimensional left G-module over a field k for a group G. For a G-module W, the dual W^* is defined as $W^* := \{f : W \to k \mid k\text{-linear}\}$, where $(g \cdot f)(*) := f(g^{-1}*)$ for $g \in G$ and $f \in W^*$. For G-modules W, W', we define the G-module $W \otimes_k W'$ by $g \cdot (u \otimes v) := gu \otimes gv$ for $g \in G$, $u \in W$, and $v \in W'$.

Lemma 4.14. *Let W, W' be finite-dimensional G-modules over a field k. Let L be a one-dimensional G-module. Suppose that $f : W \times W' \to L$ is a G-equivariant perfect pairing. In other words, the bilinear map f satisfies*

(1) *$f(u, v) = 0$ for all $v \in W' \Rightarrow u = 0$,*
(2) *$f(u, v) = 0$ for all $u \in W \Rightarrow v = 0$,*
(3) *$f(gu, gv) = g(f(u, v))$ for all $g \in G, u \in W, v \in W'$.*

Then there exists a canonical isomorphism $W' \cong W^* \otimes_k L$ as G-modules.

Proof. Let e be a non-zero vector of L. Let $\phi_e : k \to L$ be the linear isomorphism defined by $a \mapsto ae$ for $a \in k$. We define the linear map $\Phi : W' \to W^* \otimes L$ by $v \mapsto \phi_e^{-1}(f(*, v)) \otimes e$. Note that the definition of Φ is independent from the choice of e. We claim that Φ is an isomorphism as G-modules.

First, we show that Φ is G-equivariant. Let $\chi : G \to \mathrm{GL}_1(k)$ be the character associated to L. In other words, $g \cdot w = \chi(g)w$ for $g \in G$ and $w \in L$. We see that

$$\begin{aligned}\Phi(gv) &= \phi_e^{-1}(f(*, gv)) \otimes e = \phi_e^{-1}(g \cdot (f(g^{-1}*, v))) \otimes e = \phi_e^{-1}(\chi(g) f(g^{-1}*, v)) \otimes e \\ &= \phi_e^{-1}(f(g^{-1}*, v)) \otimes \chi(g)e = \phi_e^{-1}(f(g^{-1}*, v)) \otimes g \cdot e = g \cdot \Phi(v).\end{aligned}$$

Hence Φ is G-equivariant.

Next, suppose that $\Phi(v) = 0$. The assumption implies that $f(u, v) = 0$ for all $u \in W$. Because of perfectness, we have $v = 0$. Thus we proved that Φ is injective. On the other hand, we see that $\dim W' = \dim(W^* \otimes L)$, which implies that Φ is surjective. Therefore Φ is an isomorphism. □

Corollary 4.15. *Let W, W' be finite-dimensional G-modules over a field k. Let L be a one-dimensional G-module. Suppose that a bilinear map $f : W \times W' \to L$ satisfies:*

(1) *$f(u, v) = 0$ for all $v \in W' \Rightarrow u = 0$.*
(2) *$f(gu, gv) = g(f(u, v))$ for all $g \in G, u \in W, v \in W'$.*

Then there exists a canonical surjection $W' \to W^* \otimes_k L$ as G-modules.

Proof. Let $W'^{\sharp} := \{v \in W' \mid f(u,v) = 0 \text{ for all } u \in W\}$. The bilinear map $f : W \times W' \to L$ induces a G-equivariant perfect pairing $\overline{f} : W \times (W'/W'^{\sharp}) \to L$. By Lemma 4.14 we have a canonical isomorphism $\Phi : (W'/W'^{\sharp}) \cong W^* \otimes L$. Composing Φ and the projection $W' \to (W'/W'^{\sharp})$, we have a canonical surjection $W' \to W^* \otimes_k L$. □

Corollary 4.16. *Let W, W' be finite-dimensional G-modules over a field k. Let L be a one-dimensional G-module. Let Z be an irreducible G-submodule of W, and let Y be a G-submodule of W'. Suppose that any G-homomorphism $\phi : Y \to Z^* \otimes L$ is not surjective. If $f : W \times W' \to L$ is a G-equivariant perfect pairing, then $f(z,y) = 0$ for all $z \in Z, y \in Y$.*

Proof. Let $Y^{\sharp} := \{y \in Y \mid f(z,y) = 0 \text{ for all } z \in Z\}$. If $Y^{\sharp} = Y$, then the statement is true. Suppose that $Y^{\sharp} \neq Y$. Then f induces $\overline{f} : Z \times (Y/Y^{\sharp}) \to L$ which has the property that $\overline{f}(z, \overline{y}) = 0$ for all $z \in Z$ implies $\overline{y} = 0$. By Corollary 4.15, there exists a surjection $\phi : Z \to (Y/Y^{\sharp})^* \otimes L$. Since Z is irreducible, ϕ is an isomorphism. Taking $\phi \otimes L^*$ and the dual, we have $Z^* \otimes L \cong (Y/Y^{\sharp})$. Then we obtain a surjection $Y \to (Y/Y^{\sharp}) \cong Z^* \otimes L$, which is a contradiction. Hence $Y^{\sharp} = Y$. □

Proposition 4.17. *For $0 < m < n$, $r(\wedge^m(n)) = r(\wedge^{n-m}(n))$.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional irreducible representation of a group G. Assume that $\wedge^m V$ has a G-invariant realizable subspace W of dim d. We claim that $\wedge^{n-m}(V^*)$ has a G-invariant realizable subspace of dim d. Since V^* is an n-dimensional irreducible G-module, we see that $r(\wedge^m(n)) \geq r(\wedge^{n-m}(n))$ from this claim. By Changing m and $n-m$, we have $r(\wedge^{n-m}(n)) \geq r(\wedge^m(n))$, and we can conclude that $r(\wedge^m(n)) = r(\wedge^{n-m}(n))$.

Let us prove the claim. Considering the perfect pairing $\wedge^m V \times \wedge^{n-m} V \to \wedge^n V$, we have a canonical isomorphism $\Phi : \wedge^m V \cong (\wedge^{n-m}V)^* \otimes \wedge^n V$ by Lemma 4.14. Let $e_1, \dots, e_n$ be a basis of V such that $e_1 \wedge \cdots \wedge e_m \in W$. Let $f_1, \dots, f_n$ be the dual basis for $e_1, \dots, e_n$. Set $W' := \Phi(W) \otimes (\wedge^n V)^*$. Then W' is a d-dimensional G-invariant subspace of $(\wedge^{n-m}V)^* \otimes \wedge^n V \otimes (\wedge^n V)^* \cong \wedge^{n-m}(V^*)$. We easily see that W' contains $(e_1 \wedge \cdots \wedge e_m) \wedge * = f_{m+1} \wedge \cdots \wedge f_n$. This implies that $\wedge^{n-m}(V^*)$ has a G-invariant realizable subspace W' of dim d. Therefore we have proved the statement. □

Remark 4.18. By the definition, $r(\wedge^0(n)) = r(\wedge^n(n)) = 1$. Hence $r(\wedge^m(n)) = r(\wedge^{n-m}(n))$ for $0 \leq m \leq n$. We see that $r(\wedge^1(n)) = r(\wedge^{n-1}(n)) = n$ for $n \geq 2$. By Corollary 4.5 and Proposition 4.13, $2 \leq r(\wedge^m(n)) \leq n$ for $0 < m < n$. It is not easy to calculate the r-number $r(\wedge^m(n))$ in general.

Proposition 4.19. $r(\wedge^2(5)) = r(\wedge^3(5)) \geq 4$.

Proof. Since $r(\wedge^2(5)) = r(\wedge^3(5))$ by Proposition 4.17, it suffices to prove that $r(\wedge^2(5)) \geq 4$. By Proposition 4.4 we have $r(\wedge^2(5)) \geq 3$. We claim that $r(\wedge^2(5)) \neq 3$. Suppose that there exists a 3-dimensional realizable invariant subspace W of $\wedge^2 V$ for a 5-dimensional irreducible representation $\rho : G \to \mathrm{GL}(V)$ of a group G. Since W is realizable, there exist linearly independent vectors $e_1, e_2 \in V$ such that $e_1 \wedge e_2 \in W$. By irreducibility of ρ, there exists $g \in G$ such that $\rho(g)(e_1)$ can not be written as a linear combination of $\{e_1, e_2\}$. Similarly, there exists $g' \in G$ such that $\rho(g')(e_1)$ can not be written as a linear combination of $\{e_1, e_2, \rho(g)e_1, \rho(g)e_2\}$. Put $v_1 := e_1 \wedge e_2$, $v_2 := \rho(g)e_1 \wedge \rho(g)e_2$, and $v_3 := \rho(g')e_1 \wedge \rho(g')e_2$. Note that $\{e_1, e_2, \rho(g)e_1, \rho(g')e_1\}$ and $\{\rho(g)e_1, \rho(g)e_2, \rho(g')e_1\}$ are linearly independent. Then $v_1 \wedge \rho(g)e_1 \wedge \rho(g')e_1 \neq 0$ and $v_2 \wedge \rho(g')e_1 \neq 0$. We easily see that $v_1, v_2, v_3 \in W$ are linearly independent. Hence $W = \langle v_1, v_2, v_3 \rangle$.

We define the subspace $W \wedge W$ of $\wedge^4 V$ as the subspace spanned by the vectors $\{x \wedge y \in \wedge^4 V \mid x, y \in W\}$. The vector space $W \wedge W$ can be spanned by the vectors $v_1 \wedge v_2$, $v_1 \wedge v_3$, and $v_2 \wedge v_3$. Hence $W \wedge W$ is a G-invariant subspace of $\wedge^4 V$ and $\dim W \wedge W \leq 3$. Since V is irreducible, $\wedge^4 V$ is also irreducible. Thus $W \wedge W = 0$. Then there exist $g_1, g_2, g_3 \in G$ such that $\{e_1, e_2, \rho(g_1)e_1, \rho(g_2)e_1, \rho(g_3)e_1\}$ is linearly independent and $(e_1 \wedge e_2) \wedge (\rho(g_i)e_1 \wedge \rho(g_i)e_2) = 0$ for $1 \leq i \leq 3$. The vector $\rho(g_i)e_2$ can be written as a linear combination of $\{e_1, e_2, \rho(g_i)e_1\}$ for each i. So we easily see that $e_1 \wedge e_2$, $\rho(g_1)e_1 \wedge \rho(g_1)e_2$, $\rho(g_2)e_1 \wedge \rho(g_2)e_2$, and $\rho(g_3)e_1 \wedge \rho(g_3)e_2$ are linearly independent. Thus $\dim W \geq 4$. This is a contradiction. Therefore $r(\wedge^2(5)) \geq 4$. □

Later we will show that $r(\wedge^2(5)) = r(\wedge^3(5)) = 4$ in Proposition 5.3.

Proposition 4.20. $r(\wedge^2(6)) = r(\wedge^4(6)) = 3$ *and* $r(\wedge^3(6)) = 2$.

Proof. By Corollary 4.11 and Proposition 4.17, we can verify the statement. □

5. Criterion for thickness

In this section, we discuss criteria for thickness. First, we deal with 4-dimensional representations.

Proposition 5.1. *Let V be a 4-dimensional vector space over an algebraically closed field k. For a representation $\rho : G \to \mathrm{GL}(V)$, the following statements are equivalent:*

1. *ρ is thick.*
2. *ρ is 2-thick.*
3. *ρ is irreducible and the induced representation $\wedge^2\rho : G \to \mathrm{GL}(\wedge^2 V)$ has no G-invariant subspace $W \subset \wedge^2 V$ such that $2 \leq \dim W \leq 4$.*
4. *ρ is irreducible and the induced representation $\wedge^2\rho : G \to \mathrm{GL}(\wedge^2 V)$ has no G-invariant subspace $W \subset \wedge^2 V$ such that $\dim W = 2$ or 3.*

Proof. It is trivial that (1) ⇒ (2) and (3) ⇒ (4). If ρ is 2-thick, then ρ is irreducible by Proposition 2.7. Hence by Proposition 2.6, ρ is m-thick for $1 \leq m \leq 4$, which implies that (2) $\Rightarrow$ (1). Assume that ρ satisfies (4). Suppose that $\wedge^2\rho$ has a G-invariant subspace $W \subset \wedge^2 V$ of $\dim W = 4$. Then $W^\perp \subset \wedge^2 V$ is a 2-dimensional G-invariant subspace. This is a contradiction. Hence (4) $\Rightarrow$ (3).

Next, we show that (2) $\Rightarrow$ (3). Assume that ρ is 2-thick. By Proposition 2.7, ρ is irreducible. Suppose that $\wedge^2\rho$ has a non-trivial G-invariant subspace $W \subset \wedge^2 V$ such that $2 \leq \dim W \leq 4$. Put $W_1 := W$ and $W_2 := W^\perp \subset \wedge^2 V$. Then $2 \leq \dim W_2 \leq 4$. By Lemma 4.1, W_1 and W_2 are realizable. Hence it follows from Proposition 2.11 that ρ is not 2-thick. This is a contradiction. Therefore ρ satisfies (3) and we have (2) $\Rightarrow$ (3).

Finally, we show that (3) $\Rightarrow$ (2). Assume that ρ satisfies (3). Suppose that ρ is not 2-thick. It follows from Proposition 2.11 that there exist realizable invariant subspaces $W_1, W_2 \subset \wedge^2 V$ such that $W_1^\perp = W_2$. By Corollary 4.5 we have $\dim W_1 \geq 2$ and $\dim W_2 \geq 2$. Hence W_1 is a realizable invariant subspace such that $2 \leq \dim W_1 \leq 4$. This is a contradiction. Therefore (3) $\Rightarrow$ (2). We have completed the proof. □

Next, we deal with 5-dimensional representations.

Proposition 5.2. *Let V be a 5-dimensional vector space over an algebraically closed field k. For a representation $\rho : G \to \mathrm{GL}(V)$, the following are equivalent:*

(1) *ρ is thick.*
(2) *ρ is 2-thick.*
(3) *ρ is irreducible and the induced representation $\wedge^2\rho : G \to \mathrm{GL}(\wedge^2 V)$ has no non-trivial G-invariant subspace $W \subset \wedge^2 V$ with $4 \leq \dim W \leq 6$.*

Proof. It is easy to check that (1) $\Leftrightarrow$ (2) by Propositions 2.6 and 2.7. Let us show that (1) $\Rightarrow$ (3). Assume that ρ is thick. Then ρ is irreducible by Proposition 2.7. Suppose that there exists a non-trivial G-invariant subspace $W \subset \wedge^2 V$ with $4 \leq \dim W \leq 6$. Put $W_1 := W$ and $W_2 := W^\perp \subset \wedge^3 V$. By Lemma 4.1, W_1 and W_2 are realizable. Hence Proposition 2.11 implies that ρ is not thick. This is a contradiction. Therefore we see that ρ satisfies (3) and that (1) $\Rightarrow$ (3).

Finally, we show that (3) $\Rightarrow$ (2). Assume that ρ satisfies (3). Suppose that ρ is not 2-thick. Then by Proposition 2.11 there exist realizable subspaces $W_1 \subset \wedge^2 V$ and $W_2 \subset \wedge^3 V$ such that $W_1^\perp = W_2$. Proposition 4.19 says that $\dim W_1 \geq 4$ and $\dim W_2 \geq 4$. Hence $4 \leq \dim W_1 \leq 6$. This contradicts the assumption. Therefore ρ is 2-thick. □

By using Proposition 5.2, we have the following proposition.

Proposition 5.3. $r(\wedge^2(5)) = r(\wedge^3(5)) = 4$.

Proof. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of d, we denote by V_λ the irreducible representation of the symmetric group S_d over $\mathbb{C}$ corresponding to λ. Let us consider the 5-dimensional irreducible representations $V_{(3,2)}$ and $V_{(2,2,1)}$ of S_5. By calculating characters, we see that $\wedge^2 V_{(3,2)} = V_{(3,1,1)} \oplus V_{(2,1,1,1)}$ and $\wedge^2 V_{(2,2,1)} = V_{(3,1,1)} \oplus V_{(2,1,1,1)}$. Since $\dim V_{(3,1,1)} = 6$ and $\dim V_{(2,1,1,1)} = 4$, $V_{(3,2)}$ and $V_{(2,2,1)}$ are not 2-thick by Proposition 5.2. In particular, $\wedge^2 V_{(3,2)}$ and $\wedge^2 V_{(2,2,1)}$ have 4-dimensional S_5-invariant realizable subspaces $V_{(2,1,1,1)}$, respectively. This implies that $r(\wedge^2(5)) = r(\wedge^3(5)) \leq 4$. Using Proposition 4.19, we have $r(\wedge^2(5)) = r(\wedge^3(5)) = 4$. □

Remark 5.4. By the calculations above, we have the following table on $r(\wedge^m(n))$ in the case $n \leq 10$:

$n\backslash m$	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	2	4	1						
5	1	5	4	4	5	1					
6	1	6	3	2	3	6	1				
7	1	7	A	B	B	A	7	1			
8	1	8	4	C	2	C	4	8	1		
9	1	9	D	3	E	E	3	D	9	1	
10	1	10	5	F	G	2	G	F	5	10	1

Here we have not yet determined A, B, C, D, E, F, G. Until now, we can only know that $4 \leq A \leq 7$, $3 \leq B \leq 7$, $3 \leq C \leq 8$, $5 \leq D \leq 9$, $3 \leq E \leq 9$, $4 \leq F \leq 10$, and $3 \leq G \leq 10$.

For $n \geq 6$, it is difficult to check whether a given n-dimensional representation is thick or not. In the rest of this section, we show some results on thickness and denseness.

Lemma 5.5. *Let $\phi : G \to G'$ be a group homomorphism and $\rho : G' \to \mathrm{GL}(V)$ a finite-dimensional representation of G'. If ρ is not m-thick, then neither is $\rho \circ \phi : G \to \mathrm{GL}(V)$.*

Proof. Suppose that $\rho \circ \phi$ is m-thick. Let V_1 and V_2 be subspaces of V such that $\dim V_1 + \dim V_2 = \dim V$. Then there exists $g \in G$ such that $(\rho \circ \phi)(g)V_1 \oplus V_2 = V$. Putting $g' := \phi(g) \in G'$, we have $\rho(g')V_1 \oplus V_2 = V$. This implies m-thickness of ρ, which is a contradiction. Hence $\rho \circ \phi$ is not m-thick. □

Proposition 5.6. *Let k be a field. Let $V := \wedge^2 k^n$ be the exterior of the standard representation k^n of $\mathrm{GL}_n(k)$ with $n \geq 4$. Then V is not $(n-1)$-thick as a representation of $\mathrm{GL}_n(k)$.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the canonical basis of k^n. Let $W := \langle e_1 \wedge e_2, e_1 \wedge e_3, \ldots, e_1 \wedge e_n \rangle \subset \wedge^2 k^n$. Note that the $(n-1)$-dimensional subspace W is expressed as $e_1 \wedge k^n := \{e_1 \wedge v \mid v \in k^n\}$. For each $g \in \mathrm{GL}_n(k)$, $gW = \langle ge_1 \wedge ge_2, \ldots, ge_1 \wedge ge_n \rangle = (ge_1) \wedge k^n$. Put $ge_1 = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$. We see that $ge_1 \wedge e_1 \in gW$ and that

$ge_1 \wedge e_1 = -a_2 e_1 \wedge e_2 - \cdots - a_n e_1 \wedge e_n \in W$. If $ge_1 \neq a_1 e_1$, then $0 \neq ge_1 \wedge e_1 \in W \cap gW$. If $ge_1 = a_1 e_1$, then $0 \neq ge_1 \wedge ge_2 = a_1 e_1 \wedge ge_2 \in W \cap gW$. Hence $W \cap gW \neq 0$. If we choose a subspace W' of dimension $n(n-1)/2 - (n-1)$ such that $W' \supset W$, then $gW \cap W' \neq 0$ for each $g \in W$. Hence V is not $(n-1)$-thick. □

Corollary 5.7. *Let $n \geq 4$. For any n-dimensional representation V of an arbitrary group G, the exterior representation $\wedge^2 V$ of G is not $(n-1)$-thick.*

Proof. The statement follows from Lemma 5.5 and Proposition 5.6. □

In the same way as Proposition 5.6, we can prove the following proposition. This result was suggested by the referee.

Proposition 5.8. *Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation of a group G over a field k. Let $m \leq \frac{1}{2}n$. Suppose that there exists an m-dimensional subspace W of V such that $(\rho(g)W) \cap W \neq 0$ for all $g \in G$. Then ρ is not m-thick.*

Proof. Since $n-m \geq 2m-m = m$, we can choose an $(n-m)$-dimensional subspace W' of V such that $W' \supseteq W$. For all $g \in G$, $(\rho(g)W) \cap W' \supseteq (\rho(g)W) \cap W' \cap W = (\rho(g)W) \cap W \neq 0$. Hence ρ is not m-thick. □

Remark 5.9. Denseness and thickness are independent from absolute irreducibility. For example, the representation

$$\begin{array}{rccc} \rho: & \mathbb{R} & \to & \mathrm{GL}(2,\mathbb{R}) \\ & \theta & \mapsto & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{array}$$

is dense and thick, but not absolutely irreducible. Conversely, the representation $V = \wedge^2 \mathbb{C}^4$ of $\mathrm{GL}(4,\mathbb{C})$ is not thick (and hence not dense) but absolutely irreducible.

The following proposition shows that there are many examples of representations which are not dense.

Proposition 5.10. *Let $n \geq 4$. Let V be an n-dimensional irreducible representation of a group G. Assume that all irreducible representations of G have dimension $\leq n$. Then the representation V of G is not dense.*

Proof. The dimension of $\wedge^2 V$ is $\binom{n}{2} (> n)$. Hence $\wedge^2 V$ can not be irreducible. This implies that V is not dense. □

Corollary 5.11. *Let G be a finite group. Assume that G has an irreducible representation of dimension n with $n \geq 4$. Then G has an irreducible representation which is not dense.*

Proof. Let n be the maximum of the dimensions of irreducible representations of G. Since G is finite, there exists the maximum n. The assumption implies that $n \geq 4$. Let V be an irreducible representation of G of dimension n. The previous proposition shows that V is not dense. □

Remark 5.12. Let G be a group. Let V be a representation of G of dimension n with $n \geq 4$ over $\mathbb{C}$. Then $\wedge^m V$ is not thick for $2 \leq m \leq n-2$. This fact will be proven in [4].

6. Examples

In this section, we show several examples of representations for Lie groups. In [4], we will classify thick representations and dense representations for complex simple Lie groups. Here we show another approach to verify thickness and denseness.

6.1. **Case:** $G = \mathrm{GL}_2(\mathbb{C})$**.** Let a and b be integers with $a \geq 0$. Let $V_{(a+b,b)}$ be the irreducible representation of $G = \mathrm{GL}_2(\mathbb{C})$ with highest weight $(a+b, b)$. Set $\det^b := V_{(b,b)}$ for $b \in \mathbb{Z}$. Note that $V_{(a+b,b)} = \det^b \otimes V_{(a,0)}$ and that $\dim V_{(a+b,b)} = a+1$.

Lemma 6.1. *As representations of* $\mathrm{GL}_2(\mathbb{C})$*, we have*

$$\begin{aligned} \wedge^2 V_{(a+b,b)} &= \det{}^{2b} \otimes \left(\sum_{k=1}^{[\frac{a+1}{2}]} \det{}^{2k-1} \otimes V_{(2a-4k+2,0)} \right) \\ &= \sum_{k=1}^{[\frac{a+1}{2}]} V_{(2a+2b-2k+1,2b+2k-1)}. \end{aligned}$$

Proof. Comparing characters, we can verify the statement. □

Corollary 6.2. *For $a \geq 3$, the representation $V_{(a+b,b)}$ is not 2-dense. In particular, it is not dense.*

Proof. Since $[\frac{a+1}{2}] \geq 2$ if $a \geq 3$, the representation $\wedge^2 V_{(a+b,b)}$ is not irreducible by Lemma 6.1. Hence $V_{(a+b,b)}$ is not 2-dense. □

Corollary 6.3. *If $a = 3$ or 4, then the representation $V_{(a+b,b)}$ is thick.*

Proof. When $a = 3$, $\wedge^2 V_{(b+3,b)} = V_{(2b+5,2b+1)} \oplus V_{(2b+3,2b+3)}$ by Lemma 6.1. Hence $\wedge^2 V_{(3+b,b)}$ has no $\mathrm{GL}_2(\mathbb{C})$-invariant subspace W such that $2 \leq \dim W \leq 4$ because $\dim V_{(2b+5,2b+1)} = 5$ and $\dim V_{(2b+3,2b+3)} = 1$. By Proposition 5.1, the 4-dimensional representation $V_{(b+3,b)}$ is thick. When $a = 4$, $\wedge^2 V_{(b+4,b)} = V_{(2b+7,2b+1)} \oplus V_{(2b+5,2b+3)}$ by Lemma 6.1. Hence $\wedge^2 V_{(b+4,b)}$ has no $\mathrm{GL}_2(\mathbb{C})$-invariant subspace W such that $4 \leq \dim W \leq 6$ because $\dim V_{(2b+7,2b+1)} = 7$ and $\dim V_{(2b+5,2b+3)} = 3$. By Proposition 5.2, the 5-dimensional representation $V_{(b+4,b)}$ is thick. □

Remark 6.4. When $a = 1$ or 2, the representation $V_{(a+b,b)}$ is dense. When $a \geq 3$, we can verify that $V_{(a+b,b)}$ is not dense, but thick by the classification of thick representations of simple Lie groups. Indeed, we will see that $S^m\mathrm{SL}_2$ is thick and not dense if $m \geq 3$ in [4].

6.2. **Case:** $G = \mathrm{GL}_n(\mathbb{C})$**.**

Proposition 6.5. *Let* $V = \mathbb{C}^n$ *be the standard representation of* $\mathrm{GL}_n(\mathbb{C})$*. Then* V *is dense.*

Proof. This assertion follows from the irreducibility of $\wedge^i V$ for $1 \leq i \leq n-1$. $\square$

For the standard representation $V = \mathbb{C}^n$ of $\mathrm{GL}_n(\mathbb{C})$, let us discuss thickness and denseness of $\mathrm{S}^2 V$ and $\wedge^2 V$.

Lemma 6.6. *Put* $P_n(x) = \prod_{i=1}^n (1+x^i) = (1+x)(1+x^2)\cdots(1+x^n)$*. Let* a_i *be the coefficient of* x^i *in* $P_n(x)$*. Then if* $n \geq 3$*, then* $a_i \geq 1$ *for any* $0 \leq i \leq \frac{n(n+1)}{2}$ *and* $a_i \geq 2$ *for any* $3 \leq i \leq \frac{n(n+1)}{2} - 3$*.*

Proof. Let us prove the statement by induction on n. When $n = 3$, $P_3(x) = 1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6$ and hence the statement holds. Suppose that the statement is true for n. Since

$$\begin{aligned}
P_{n+1}(x) &= P_n(x)(1+x^{n+1}) \\
&= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + (a_{n+1} + a_0)x^{n+1} + (a_{n+2} + a_1)x^{n+2} \\
&\quad + \cdots + (a_{n(n+1)/2} + a_{(n+1)(n-2)/2})x^{n(n+1)/2} + a_{n(n-1)/2}x^{(n^2+n+2)/2} \\
&\qquad\qquad + \cdots + a_{n(n+1)/2}x^{(n+1)(n+2)/2},
\end{aligned}$$

the statement is also true for $n+1$. This completes the proof. $\square$

Proposition 6.7. *Let* $V = \mathbb{C}^n$ *be the standard representation of* $\mathrm{GL}_n(\mathbb{C})$ $(n \geq 3)$*. Then the second symmetric tensor* $\mathrm{S}^2 V$ *is irreducible, but not* m*-thick for* $3 \leq m \leq \frac{n(n+1)}{2} - 3$*.*

Proof. It is well-known that $\mathrm{S}^2 V$ is irreducible. By [2, Theorem 4.4.2], the number of irreducible components of $\wedge^m(\mathrm{S}^2 V)$ is equal to the the number of partitions of m into distinct parts of size at most n. This number is equal to the coefficient a_m of x^m in $P_n(x)$ in Lemma 6.6. Since $a_m \geq 2$ by Lemma 6.6, $\wedge^m(\mathrm{S}^2 V)$ is not irreducible for any $3 \leq m \leq \frac{n(n+1)}{2} - 3$. We see that irreducible components of $\wedge^m(\mathrm{S}^2 V)$ are all realizable by the proof of [2, Theorem 4.4.2]. Hence Proposition 2.11 implies that $\wedge^m(\mathrm{S}^2 V)$ is not m-thick for any $3 \leq m \leq \frac{n(n+1)}{2} - 3$. $\square$

Proposition 5.6 shows that $\wedge^2 \mathbb{C}^n$ is not $(n-1)$-thick for the standard representation $\mathbb{C}^n$ of $\mathrm{GL}_n(\mathbb{C})$ for $n \geq 4$. Moreover, we have the following proposition.

Proposition 6.8. *Let* $V = \mathbb{C}^n$ *be the standard representation of* $\mathrm{GL}_n(\mathbb{C})$ $(n \geq 4)$*. Then the second alternating tensor* $\wedge^2 V$ *is irreducible, but not* m*-thick for* $3 \leq m \leq \frac{n(n-1)}{2} - 3$*.*

Proof. It is well-known that $\wedge^2 V$ is irreducible. By [2, Theorem 4.4.4], the number of irreducible components of $\wedge^m(\wedge^2 V)$ is equal to the the number of partitions of m into distinct parts of size at most $n-1$. This number is equal to the coefficient a_m of x^m in $P_{n-1}(x)$ in Lemma 6.6. Since $a_m \geq 2$ by Lemma 6.6, $\wedge^m(\wedge^2 V)$ is not irreducible for any $3 \leq m \leq \frac{n(n-1)}{2} - 3$. We see that irreducible components of $\wedge^m(\wedge^2 V)$ are all realizable by the proof of [2, Theorem 4.4.4]. Hence Proposition 2.11 implies that $\wedge^m(\wedge^2 V)$ is not m-thick for any $3 \leq m \leq \frac{n(n-1)}{2} - 3$. □

Then from Propositions 6.7 and 6.8, we have the following corollary.

Corollary 6.9. *Let $V = \mathbb{C}^n$ be an n-dimensional representation of any group G. If $n \geq 3$, then the second symmetric tensor $\mathrm{S}^2 V$ is not thick. If $n \geq 4$, then the second alternating tensor $\wedge^2 V$ is not thick.*

Proof. Using Lemma 5.5, we can prove the statement (the latter part has been proved in Corollary 5.7). □

6.3. **Case:** $G = \mathrm{SO}_n(\mathbb{C})$**.**

Proposition 6.10. *Let V be the standard representation of $G = \mathrm{SO}_{2n}(\mathbb{C})$. Then V is m-dense for each $0 < m < 2n$ with $m \neq n$, but not n-thick.*

Proof. The first assertion follows from the irreducibility of $\wedge^i V$ for $1 \leq i \leq n-1$. The proof of [1, Theorem 19.2] shows that the n-th alternating tensor $\wedge^n V$ has exactly two irreducible factors and they are realizable. Then by Proposition 2.11 the representation V is not n-thick. □

Proposition 6.11. *Let V be the standard representation of $G = \mathrm{SO}_{2n+1}(\mathbb{C})$. Then V is dense.*

Proof. The m-th alternating tensor $\wedge^m V$ is irreducible for each $0 < m < 2n+1$ (for example, see [1, Theorem 19.14]). This implies the statement. □

6.4. **Case:** $G = \mathrm{Sp}_{2n}(\mathbb{C})$**.** Let V be a $2n$-dimensional complex vector space, $\{e_1, e_2, \ldots, e_{2n}\}$ a basis for V, and $\{e_1^*, e_2^*, \ldots, e_{2n}^*\}$ its dual basis for the dual vector space V^*. We use a non-degenerate skew-symmetric bilinear form $\omega = \displaystyle\sum_{i=1}^{n} e_i^* \wedge e_{n+i}^*$ and the corresponding symplectic Lie group $\mathrm{Sp}_{2n}(\mathbb{C})$. Then we have a contraction map by ω:

$$f_m : \wedge^m V \to \wedge^{m-2} V.$$

If $m \leq n$, $\mathrm{Ker} f_m$ is the m-th fundamental representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. In particular, $\mathrm{Ker} f_m$ is irreducible. We have the isotropic Grassmann variety of isotropic subspaces of dimension m as a unique minimal closed orbit in the projective space $\mathbf{P}(\mathrm{Ker} f_m)$. Since $\mathrm{Ker} f_m$ contains $\wedge^m L$ for any isotropic m-dimension subspace $L \subset V$, $\mathrm{Ker} f_m$ is realizable. For details see [1].

The following lemma is well-known.

Lemma 6.12. *Let (V,ω) be a $2n$-dimensional symplectic vector space and $W \subset V$ a subspace. Then there is a basis $\{v_1, v_2, \dots, v_{2n}\}$ of V such that $\omega(v_i, v_{n+i}) = 1$, $\omega(v_i, v_j) = 0$ if $j \neq i \pm n$, and for some non-negative integers l, k*

$$W = \langle v_1, \dots, v_k, v_{k+1}, \dots, v_{n-l}, v_{n+1}, \dots, v_{n+k} \rangle.$$

Lemma 6.13. *Let (V,ω) be a $2n$-dimensional symplectic vector space and $W \subset V$ a subspace of codimension i $(i \leq n)$. Then there is a Lagrangian subspace $L \subset V$ such that $L + W = V$.*

Proof. It is enough to prove the case of $i = n$. By Lemma 6.12, there is a symplectic basis $\{v_1, v_2, \dots, v_{2n}\}$ of V such that for some non-negative integer $k \leq \frac{n}{2}$

$$W = \langle v_1, \dots, v_k, v_{k+1}, \dots, v_{n-k}, v_{n+1}, \dots, v_{n+k} \rangle.$$

In the case of $k = 0$, $W = \langle v_1, \dots, v_n \rangle$. Then $L = \langle v_{n+1}, \dots, v_{2n} \rangle$ satisfies the condition $L + W = V$. In the case of $1 \leq k \leq \frac{n}{2}$, we put as following,

$$\begin{aligned} L = \langle & v_{n+k+1}, \dots, v_{n+k+i}, \dots, v_{2n-k}, \\ & v_{n-k+1} + v_{n+1}, \dots, v_{n-k+i} + v_{n+i}, \dots, v_n + v_{n+k}, \\ & v_{2n-k+1} + v_1, \dots, v_{2n-k+i} + v_i, \dots, v_{2n} + v_k \rangle. \end{aligned}$$

Then L is a Lagrangian subspace and satisfies the condition $L + W = V$. □

Lemma 6.14. *Let (V,ω) be a $2n$-dimensional symplectic vector space and $W \subset V$ a subspace of codimension i $(i \leq n)$. Then there is an isotropic subspace $U \subset V$ of dimension i such that $U \cap W = \{0\}$.*

Proof. By Lemma 6.13, there is a Lagrangian subspace $L \subset V$ such that $L + W = V$. Since the dimension of $L \cap W$ is $n - i$, there is a subspace $U \subset L$ such that the dimension of U is i and $U \cap W = \{0\}$. Since U is a subspace of a Lagrangian subspace L, U is an isotropic subspace. □

Then we have the following proposition.

Proposition 6.15. *Let (V,ω) be the standard representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. For each $1 < m \leq n$, $(\mathrm{Ker} f_m)^\perp \subset \wedge^{2n-m} V$ is not realizable.*

Proof. If $(\mathrm{Ker} f_m)^\perp$ is realizable, there is a subspace $W \subset V$ of codimension m such that $\wedge^{2n-m} W \in (\mathrm{Ker} f_m)^\perp$. Then by Lemma 6.14 we have an isotropic subspace $U \subset V$ of dimension m such that $U \cap W = \{0\}$. Because $\mathrm{Ker} f_m$ contains $\wedge^m L$ for any isotropic subspace $L \subset V$ of dimension m, we have $\wedge^m U \in \mathrm{Ker} f_m$. But we have $(\wedge^{2n-m} W) \wedge (\wedge^m U) \neq 0$. This is a contradiction. □

By the $\mathrm{SL}_{2n}(\mathbb{C})$-equivariant canonical pairing $\wedge^{2n-k} V \times \wedge^k V \to \wedge^{2n} V \cong \mathbb{C}$, we have the $\mathrm{SL}_{2n}(\mathbb{C})$-equivariant isomorphism

$$\wedge^{2n-k} V \to (\wedge^k V)^* \cong \wedge^k V^*.$$

Moreover by the correspondence $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \mapsto e_{i_1} \wedge \cdots \wedge e_{i_k}$ we have the isomorphism $\wedge^k V^* \to \wedge^k V$ as vector spaces. The difference between these vector spaces as $\mathrm{SL}_{2n}(\mathbb{C})$-modules is described by the outer automorphism

$$\begin{array}{cccc} \sigma & : & \mathrm{SL}_{2n}(\mathbb{C}) & \to & \mathrm{SL}_{2n}(\mathbb{C}) \\ & & g & \mapsto & {}^t g^{-1}. \end{array}$$

Then we obtain the isomorphism ϕ as $\mathrm{SL}_{2n}(\mathbb{C})$-modules up to the outer automorphism σ, that is,

$$\phi : \wedge^{2n-k} V \to (\wedge^k V)^* \cong \wedge^k V^* \to \wedge^k V.$$

Thereby ϕ induces the isomorphism $\overline{\phi}$ as follows:

$$\begin{array}{ccccc} \overline{\phi} & : & \mathbb{P}(\wedge^{2n-k} V) & \to & \mathbb{P}(\wedge^k V) \\ & & \cup & & \cup \\ & & \mathrm{Gr}(2n-k, V) & \cong & \mathrm{Gr}(k, V). \end{array}$$

Since $\sigma(\mathrm{Sp}_{2n}(\mathbb{C})) = \mathrm{Sp}_{2n}(\mathbb{C})$ and σ is an inner automorphism of $\mathrm{Sp}_{2n}(\mathbb{C})$, ϕ gives an isomorphism between $\wedge^{2n-k} V$ and $\wedge^k V$ as $\mathrm{Sp}_{2n}(\mathbb{C})$-modules. When we consider $\wedge^k V$ as a $\mathrm{Sp}_{2n}(\mathbb{C})$-module, it is well-known that each irreducible representation of $\mathrm{Sp}_{2n}(\mathbb{C})$ occurs at most once in an irreducible decomposition of $\wedge^k V$ (see [1, Chap.17]). Then we have several irreducible $\mathrm{Sp}_{2n}(\mathbb{C})$-invariant subspaces $\{W_i\}_{i=1,\ldots,s}$ in $\wedge^k V$ such that we have a unique irreducible decomposition $\wedge^k V = W_1 \oplus W_2 \oplus \cdots \oplus W_s$, and $W_i \cong W_j$ if and only if $i = j$. Since there exists some number i such that $W_i = \mathrm{Ker} f_k$, from now we put $W_1 = \mathrm{Ker} f_k$. Therefore under the isomorphism ϕ we can obtain the unique irreducible decomposition of $\wedge^{2n-k} V$. Namely if we put $W_i' := \phi^{-1}(W_i)$, $\{W_i'\}_{i=1,\ldots,s}$ are $\mathrm{Sp}_{2n}(\mathbb{C})$-invariant subspaces in $\wedge^{2n-k} V$ such that we have the unique irreducible decomposition $\wedge^{2n-k} V = W_1' \oplus W_2' \oplus \cdots \oplus W_s'$, and $W_i' \cong W_j'$ if and only if $i = j$. By the above construction we have the following lemma.

Lemma 6.16. *For any subset $\{j_1, \ldots, j_l\} \subset \{1, 2, \ldots, s\}$, the subset $\mathbb{P}(W_{j_1} \oplus \cdots \oplus W_{j_l}) \cap \mathrm{G}r(k, V)$ is empty if and only if the subset $\mathbb{P}(W_{j_1}' \oplus \cdots \oplus W_{j_l}') \cap \mathrm{G}r(2n-k, V)$ is empty.*

Proposition 6.17. *For any subset $\{j_1, \ldots, j_l\} \subset \{1, 2, \ldots, s\}$, the following are equivalent:*

(1) *$W_{j_1} \oplus \cdots \oplus W_{j_l}$ is a realizable subspace of $\wedge^k V$.*
(2) *$W_{j_1}' \oplus \cdots \oplus W_{j_l}'$ is a realizable subspace of $\wedge^{2n-k} V$.*
(3) *There is some $m \in \{1, \ldots, l\}$ such that $j_m = 1$.*

Proof. Lemma 6.16 shows that (1) and (2) are equivalent. Note that $W^* \cong W$ for any $\mathrm{Sp}_{2n}(\mathbb{C})$-modules W. For the perfect paring $\wedge^k V \times \wedge^{2n-k} V \to \wedge^{2n} V \cong k$, we see that $(\mathrm{Ker} f_k)^\perp = W_2' \oplus W_3' \oplus \cdots \oplus W_s'$. Indeed, any $\mathrm{Sp}_{2n}(\mathbb{C})$-homomorphism $\phi : W_1 = \mathrm{Ker} f_k \to (W_2' \oplus W_3' \oplus \cdots \oplus W_s')^* \cong W_2' \oplus W_3' \oplus \cdots \oplus W_s'$ is zero. By Corollary 4.16, we have $(\mathrm{Ker} f_k)^\perp \supseteq W_2' \oplus W_3' \oplus \cdots \oplus W_s'$. Since $W_1' \cong W_1$ is irreducible, $(\mathrm{Ker} f_k)^\perp = W'_2 \oplus W'_3 \oplus \cdots \oplus W'_s$. Then Proposition 6.15 shows that (2) and (3) are equivalent. □

Then we have the following proposition.

Proposition 6.18. *The standard representation of* $\mathrm{Sp}_{2n}(\mathbb{C})$ *is thick, but not* m*-dense for each* $1 < m < 2n - 1$.

Proof. Since each irreducible representation occurs at most once in $\wedge^k V$, for any invariant subspace $U \subset \wedge^k V$ there is a subset $\{i_1, \ldots, i_\alpha\} \subset \{1, 2, \ldots, s\}$ such that $U = W_{i_1} \oplus \cdots \oplus W_{i_\alpha}$. Similarly for $U^\perp$ there is a subset $\{j_1, \ldots, j_\beta\} \subset \{1, 2, \ldots, s\}$ such that $U^\perp = W'_{j_1} \oplus \cdots \oplus W'_{j_\beta}$. Since $(\mathrm{Ker} f_k)^\perp = W'_2 \oplus W'_3 \oplus \cdots \oplus W'_s$, $1 \in \{i_1, \ldots, i_\alpha\}$ if and only if $1 \notin \{j_1, \ldots, j_\beta\}$. By Proposition 6.17, it is impossible that both U and $U^\perp$ are realizable. This implies that V is thick. Since it is well-known that $\wedge^m V$ is not irreducible for each $1 < m < 2n - 1$, V is not m-dense for $1 < m < 2n - 1$. □

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