THE CLASSIFICATION OF THICK REPRESENTATIONS OF SIMPLE LIE GROUPS

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Abstract. We characterize finite-dimensional thick representations over $\mathbb{C}$ of connected complex semi-simple Lie groups by irreducible representations which are weight multiplicity-free and whose weight posets are totally ordered sets. Moreover, using this characterization, we give the classification of thick representations over $\mathbb{C}$ of connected complex simple Lie groups.

1. Introduction

In our previous paper [7], we have introduced m-thickness and thickness of group representations. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of a group G. If for any subspaces V_1 and V_2 of V with $\dim V_1 = m$ and $\dim V_2 = \dim V - m$ there exists $g \in G$ such that $(\rho(g)V_1) \oplus V_2 = V$, we say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-thick*. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *thick* if ρ is m-thick for each $0 < m < \dim V$ (Definition 2.1). Remark that 1-thickness is equivalent to irreducibility (Proposition 2.8). Hence m-thickness is a natural generalization of irreducibility of group representations.

Let G be a connected semi-simple Lie group over $\mathbb{C}$, B a Borel subgroup of G, T a maximal torus which is contained in B. Denote their Lie algebras by $\mathfrak{g}, \mathfrak{b}$ and $\mathfrak{t}$, respectively. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional irreducible representation of G over $\mathbb{C}$. We denote the set of $\mathfrak{t}$-weights in V by $W(V)$. Choosing a set of simple roots for $(\mathfrak{g}, \mathfrak{t})$, we can regard $W(V)$ as a partially ordered set (poset) with respect to the usual root order. We call it the *weight poset*. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *weight multiplicity-free* if the weight spaces in V are all one-dimensional. We give the following characterization of thickness.

Theorem 1.1 (Theorem 3.5). *An irreducible representation $\rho : G \to \mathrm{GL}(V)$ of a connected semi-simple Lie group G is thick if and only if it is weight multiplicity-free and its weight poset is a totally ordered set.*

Using this characterization, we can classify the complex thick representations of connected semi-simple Lie groups.

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Theorem 1.2 (Theorems 3.11 and 3.12)**.** *If a representation of a connected semi-simple Lie group is thick, then it is geometrically equivalent to one of the following list:*

$$e,\ \mathrm{SL}_n(n \geq 2),\ S^m\mathrm{SL}_2(m \geq 2),\ \mathrm{SO}_{2n+1}(n \geq 2),\ \mathrm{Sp}_{2n}(n \geq 2),\ \mathrm{G}_2.$$

Here the irreducible representation of a connected simple Lie group G of the highest weight ω_1, where ω_1 is the first fundamental weight, is denoted by G. Similarly, S^mG stands for the m-th symmetric power of G. Let e denote the trivial 1-dimensional representation for any group G. For the definition of geometric equivalence, see Definition 3.8.

We denote by ω_i the i-th fundamental weight for a connected simple Lie group G. In §3, all Lie groups are assumed to be over $\mathbb{C}$ and all representations are finite-dimensional over $\mathbb{C}$.

2. Preliminaries

A *representation* of a group G on a vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. Then such a map ρ gives V the structure of a G-module. We sometimes call V itself a representation of G and write gv for $\rho(g)(v)$. We recall several definitions and results in our previous paper [7].

Definition 2.1 ([7, Definition 2.1])**.** Let G be a group. Let V be a finite-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-thick* if for any subspaces V_1 and V_2 of V with $\dim V_1 = m$ and $\dim V_2 = \dim V - m$, there exists $g \in G$ such that $(\rho(g)V_1) \oplus V_2 = V$. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *thick* if ρ is m-thick for each $0 < m < \dim V$.

Definition 2.2 ([7, Definition 2.3])**.** Let G be a group. Let V be a finite-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-dense* if the induced representation $\wedge^m \rho : G \to \mathrm{GL}(\wedge^m V)$ is irreducible. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *dense* if ρ is m-dense for each $0 < m < \dim V$.

We show several examples. See [7] for details.

Example 2.3 (*cf.* [7, Proposition 6.5])**.** Let V be the standard representation of SL_n and V^* the dual representation of V. Then V and V^* are dense.

Example 2.4 ([7, Proposition 6.10])**.** The standard representation of SO_{2n} is m-dense for each $0 < m < 2n$ with $m \neq n$, but not n-thick.

Example 2.5 ([7, Proposition 6.11])**.** The standard representation of SO_{2n+1} is dense.

Example 2.6 ([7, Proposition 6.18])**.** The standard representation of Sp_{2n} is thick, but not m-dense for each $1 < m < 2n - 1$.

Let V be a finite-dimensional representation of a group G. For positive integers i and j with $i + j = \dim V$, let us consider the G-equivariant perfect pairing $\wedge^i V \otimes \wedge^j V \xrightarrow{\wedge} \wedge^{\dim V} V \cong k$. For a G-invariant subspace W of $\wedge^i V$, put $W^\perp := \{y \in \wedge^j V \mid x \wedge y = 0 \text{ for any } x \in W\}$. Then $W^\perp$ is also G-invariant. In particular, $\wedge^i V$ is irreducible if and only if so is $\wedge^j V$.

Proposition 2.7 ([7, Proposition 2.6]). *Let V be an n-dimensional representation of a group G. For each $0 < m < n$, V is m-thick (resp. m-dense) if and only if V is $(n-m)$-thick (resp. $(n-m)$-dense).*

Proposition 2.8 ([7, Proposition 2.7]). *For any finite-dimensional representation V of a group G, the following implications hold for $0 < m < \dim V$:*

$$\begin{array}{ccccc} m\text{-dense} & \Longrightarrow & m\text{-thick} & & \\ & & \Downarrow & & \\ 1\text{-dense} & \Longleftrightarrow & 1\text{-thick} & \Longleftrightarrow & \text{irreducible.} \end{array}$$

Corollary 2.9 ([7, Corollary 2.8]). *For any finite-dimensional representation of a group G, the following implications hold*:

$$dense \Rightarrow thick \Rightarrow irreducible.$$

Corollary 2.10 ([7, Corollary 2.9]). *For any representation V of a group G with $\dim V \leq 3$, the following implications hold*:

$$dense \Leftrightarrow thick \Leftrightarrow irreducible.$$

Definition 2.11 ([7, Definition 2.10]). Let V be an n-dimensional vector space over a field k. For a d-dimensional subspace V' of V with $0 < d < n$, we can consider a point $[\wedge^d V']$ in the projective space $\mathbb{P}(\wedge^d V)$. In the sequel, we identify $[\wedge^d V']$ with a non-zero vector $\wedge^d V' \in \wedge^d V$ (which is determined by $[\wedge^d V']$ up to scalar) for simplicity. For a vector subspace $W \subset \wedge^d V$, we say that W is *realizable* if W contains a non-zero vector $\wedge^d V'$ obtained by a d-dimensional subspace V' of V.

We have the following criterion of thickness.

Proposition 2.12 ([7, Proposition 2.11]). *Let V be an n-dimensional representation of a group G. For $0 < m < n$, V is not m-thick if and only if there exist G-invariant realizable subspaces $W_1 \subseteq \wedge^m V$ and $W_2 \subseteq \wedge^{n-m} V$ such that $W_1^\perp = W_2$.*

3. The classification of thick representations of simple Lie groups

Let G be a connected semi-simple Lie group over the complex number field $\mathbb{C}$, B a Borel subgroup of G, T a maximal torus which is contained in B, B^- a Borel subgroup of G opposite to B relative to $T = B \cap B^-$. Denote their Lie algebras by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ and $\mathfrak{b}^-$, respectively. Let V be a finite-dimensional irreducible representation of G over $\mathbb{C}$. We will denote the set of $\mathfrak{t}$-weights in V by $W(V)$. For any weight $\varphi \in W(V)$, let V_φ be the φ-weight space in V. Let Π be the set of simple roots and Δ^+ the set of positive roots for $(\mathfrak{g}, \mathfrak{b})$. We can regard $W(V)$ as a partially ordered set (poset) with respect to the usual root order. More precisely, $\mu > \gamma$ if and only if $\mu - \gamma$ is a nonzero sum of simple roots with nonnegative coefficients. In particular, if $\mu - \gamma$ is a simple root, we say that μ *covers* γ. We call $W(V)$ the *weight poset*. We say that a representation V of G is *weight multiplicity-free* (*WMF*) if the weight spaces in V are all one-dimensional. Howe [3] classified the irreducible representations of connected simple Lie groups which are weight multiplicity-free.

Proposition 3.1. *If a representation V of G is thick, it is weight multiplicity-free.*

Proof. Assume that V is not WMF. Then there exists a weight $\varphi \in W(V)$ such that the dimension of V_φ is larger than one. Let $W^+(\varphi)$ be the set of all weights strictly larger than φ, and $Y^+(\varphi)$ the subspace which is spanned by all weight spaces for weights in $W^+(\varphi)$. Because the dimension of V_φ is larger than one, we can choose two linear independent φ-weight vectors v and w. Let $W^{a,b}_\varphi(+)$ be $\mathbb{C}(av + bw) \oplus Y^+(\varphi)$ for $a, b \in \mathbb{C}$. The subspace $W^{a,b}_\varphi(+)$ is B-invariant. Let n be the dimension of V, and d the dimension of $W^{a,b}_\varphi(+)$ for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. The elements $\wedge^d W^{a,b}_\varphi(+)$ for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ are distinct B-eigenvectors in $\wedge^d V$ with the same weight. Let $U^{a,b}_\varphi(+)$ be the irreducible G-submodule in $\wedge^d V$ with the highest weight vector $\wedge^d W^{a,b}_\varphi(+)$. Let $U_\varphi(+)$ be the direct sum $U^{1,0}_\varphi(+) \oplus U^{0,1}_\varphi(+) \subset \wedge^d V$. Any irreducible G-submodule of $U_\varphi(+)$ is equal to $U^{a,b}_\varphi(+)$ for some $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Hence any irreducible G-submodule of $U_\varphi(+)$ is realizable.

Let $Y^-(\varphi)$ be the subspace which is spanned by all weight spaces for weights in $W(V) \setminus \{W^+(\varphi), \varphi\}$. We take a basis $\{v, w, u_1, \ldots, u_s\}$ for V_φ which contains v, w. Let $W_\varphi(-)$ be the subspace which is spanned by $\{w, u_1, \ldots, u_s\}$ and $Y^-(\varphi)$. The subspace $W_\varphi(-)$ is invariant under the action of the opposite Borel subgroup B^-. The equalities $\dim W_\varphi(-) = \dim V - \dim W^{a,b}_\varphi(+) = n - d$ hold for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then $\wedge^{n-d} W_\varphi(-)$ is a B^--eigenvector in $\wedge^{n-d} V$. Let $U_\varphi(-)$ be the irreducible G-submodule with the lowest weight vector $\wedge^{n-d} W_\varphi(-)$ in $\wedge^{n-d} V$. Obviously, $U_\varphi(-)$ is realizable. The irreducibility of $U_\varphi(-)$ shows the irreducibility of $\wedge^d V/(U_\varphi(-))^\perp$. Then $(U_\varphi(-))^\perp \cap U_\varphi(+) \neq \{0\}$ because $U_\varphi(+)$ is not irreducible. Hence $(U_\varphi(-))^\perp \cap U_\varphi(+)$ contains some realizable G-submodule $U^{a,b}_\varphi(+)$. Therefore $(U_\varphi(-))^\perp$ is realizable. Putting $W_1 = (U_\varphi(-))^\perp$ and $W_2 = U_\varphi(-)$, we see that V is not thick by Proposition 2.12. Hence if V is thick, then it is WMF. □

Proposition 3.2. *If a representation V of G is thick, its weight poset $W(V)$ is a totally ordered set.*

Proof. Let V be a thick representation of G. By Proposition 3.1, V is WMF. For any weight $\phi \in W(V)$, let $W^+(\phi)$ be the set of all weights strictly larger than ϕ, and $Y^+(\phi)$ the subspace which is spanned by all weight spaces for weights in $W^+(\phi)$. Note that the irreducible representation V has a highest weight ω and that each weight of V has the form $\omega - \sum_{i=1}^{l} m_i\alpha_i$ $(m_i \in \mathbb{N})$, where $\Pi = \{\alpha_1, \ldots, \alpha_l\}$.

Suppose that the weight poset $W(V)$ is not a totally ordered set. There exists a positive integer i such that $W(V)$ has the i-th highest weight, but not the $(i+1)$-th highest weight. Let φ be the i-th highest weight, and ψ_1, ψ_2 maximal weights in $W(V) \setminus (W^+(\varphi) \cup \{\varphi\})$. Then the subset $W^+(\varphi) \cup \{\varphi\}$ is a totally ordered set, φ covers ψ_1, ψ_2, and $W^+(\psi_1) = W^+(\psi_2) = W^+(\varphi) \cup \{\varphi\}$. Because V is WMF, there exists a unique ψ_i-weight vector v_i up to scalar for each $i = 1, 2$. Let $W_{\psi_i}(+)$ be $\mathbb{C} \cdot v_i \oplus Y^+(\psi_i)$. The subspaces $W_{\psi_i}(+)$ are B-invariant for each $i = 1, 2$. Let n be the dimension of V, and d the dimension of $W_{\psi_i}(+)$. The elements $\wedge^d W_{\psi_1}(+)$ and $\wedge^d W_{\psi_2}(+)$ are distinct B-eigenvectors with distinct weights in $\wedge^d V$. Let $U_{\psi_i}(+)$ be the irreducible G-submodule of $\wedge^d V$ with the highest weight vector $\wedge^d W_{\psi_i}(+)$ for each $i = 1, 2$. Then $U_{\psi_1}(+)$ and $U_{\psi_2}(+)$ are realizable and not isomorphic to each other as G-modules. Let $Y^-(\psi_1)$ be the subspace which is spanned by all weight spaces for weights in $W(V) \setminus (W^+(\psi_1) \cup \{\psi_1\})$. The subspace $Y^-(\psi_1)$ is invariant under the action of the opposite Borel subgroup B^-. The equalities $\dim Y^-(\psi_1) = \dim V - \dim W_{\psi_1}(+) = n - d$ hold. Then $\wedge^{n-d} Y^-(\psi_1)$ is a B^--eigenvector in $\wedge^{n-d} V$. Let $U_{\psi_1}(-)$ be the irreducible G-submodule of $\wedge^{n-d} V$ with the lowest weight vector $\wedge^{n-d} Y^-(\psi_1)$. Then $U_{\psi_1}(-)$ is realizable. The irreducibility of $U_{\psi_1}(-)$ shows the irreducibility of $\wedge^d V / (U_{\psi_1}(-))^\perp$. Then $(U_{\psi_1}(-))^\perp \cap (U_{\psi_1}(+) \oplus U_{\psi_2}(+)) \neq \{0\}$. Because $U_{\psi_1}(+)$ is not isomorphic to $U_{\psi_2}(+)$, $U_{\psi_1}(+) \subset (U_{\psi_1}(-))^\perp$ or $U_{\psi_2}(+) \subset (U_{\psi_1}(-))^\perp$. In particular, $(U_{\psi_1}(-))^\perp$ is realizable. Putting $W_1 = (U_{\psi_1}(-))^\perp$ and $W_2 = U_{\psi_1}(-)$, we see that V is not thick by Proposition 2.12. This is a contradiction. Hence $W(V)$ is a totally ordered set. $\square$

Let us denote the Grassmann variety which is the set of all k-dimensional subspaces of a vector space V by $\mathrm{Grass}(k, V) (\subset \mathbb{P}(\wedge^k V))$.

Lemma 3.3. *Let V be a representation of G, and W a G-invariant realizable subspace of $\wedge^k V$. Then there exists $[v] \in \mathbb{P}(W) \cap \mathrm{Grass}(k, V)$ such that $[v]$ is B-invariant.*

Proof. Let X be $\mathbb{P}(W) \cap \mathrm{Grass}(k, V)$. Because W is realizable, X is not empty. Note that X is G-invariant and compact. We take a G-orbit O in X whose dimension is minimal. The orbit O is closed and then compact. There is a parabolic subgroup P of G such that the orbit O is isomorphic to G/P. Then there is a point $[v] \in O \subset \mathbb{P}(W) \cap \mathrm{Grass}(k, V)$ such that $[v]$ is B-invariant. $\square$

Lemma 3.4. *Assume that an irreducible representation V of G is weight multiplicity-free, its weight poset $W(V)$ is a totally ordered set $\{\varphi_1 > \varphi_2 > \cdots > \varphi_n\}$, and W is a G-invariant realizable subspace of $\wedge^k V$. Let v_i be a nonzero vector in the φ_i-weight space V_{φ_i} $(i = 1, 2, \ldots, n)$. Then W contains $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ and $v_{n-(k-1)} \wedge v_{n-(k-2)} \wedge \cdots \wedge v_n$.*

Proof. Because V is weight multiplicity-free, $\{v_1, \ldots, v_n\}$ is a basis of V. By Lemma 3.3, there exists $[v] \in \mathbb{P}(W) \cap \mathrm{Grass}(k, V)$ such that v is a highest weight vector of an irreducible subrepresentation of W with respect to B. We can put

$$\begin{aligned} v = \quad & (p_{1,1}v_1 + p_{1,2}v_2 + \cdots + p_{1,n}v_n) \\ & \wedge(p_{2,1}v_1 + p_{2,2}v_2 + \cdots + p_{2,n}v_n) \\ & \vdots \\ & \wedge(p_{k,1}v_1 + p_{k,2}v_2 + \cdots + p_{k,n}v_n) \end{aligned}$$

up to scalar multiplication, where $P = (p_{i,j})$ is in reduced row echelon form. Remark that P is uniquely determined. Let X_α be a root vector for a positive root $\alpha \in \Delta^+$. Then $X_\alpha v = 0$ holds for any $\alpha \in \Delta^+$. If $p_{1,1} = p_{1,2} = \cdots = p_{1,i} = 0$ and $p_{1,i+1} = 1$ for $i \geq 1$, there is a positive root $\alpha \in \Delta^+$ such that $X_\alpha v_{i+1}$ is cv_i for a nonzero constant c. Then $X_\alpha v$ is not 0. This is a contradiction. So $p_{1,1} = 1$. Similarly, we can show that $p_{22} = \cdots = p_{kk} = 1$. Because v is a highest weight vector, for any $t \in \mathfrak{t}$ there is a constant c such that $tv = cv$. Then by the uniqueness of P we can show that $p_{ij} = 0$ for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$. Then $v = v_1 \wedge v_2 \wedge \cdots \wedge v_k \in W$. A similar argument with respect to B^- shows that $v_{n-(k-1)} \wedge v_{n-(k-2)} \wedge \cdots \wedge v_n \in W$. $\square$

Theorem 3.5. *An irreducible representation V of a connected semi-simple Lie group G is thick if and only if it is weight multiplicity-free and its weight poset is a totally ordered set.*

Proof. The "only if" part can be proved by Propositions 3.1 and 3.2. Let us prove the "if" part. Let us use the notations in Lemma 3.4. Assume that $W_1 \subseteq \wedge^k V$ and $W_2 \subseteq \wedge^{n-k} V$ are G-invariant realizable subspaces. By Lemma 3.4, $v_1 \wedge v_2 \wedge \cdots \wedge v_k \in W_1$ and $v_{k+1} \wedge v_{k+2} \wedge \cdots \wedge v_n \in W_2$. Since $(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \wedge (v_{k+1} \wedge v_{k+2} \wedge \cdots \wedge v_n) \neq 0$, $W_1^\perp \neq W_2$. By Proposition 2.12, V is thick. $\square$

By [3, Theorem 4.6.3], we have Howe's classification of irreducible representations of connected simple Lie groups which are weight multiplicity-free. We also refer to Panyushev's paper [8, Table 1] for the weight posets of weight multiplicity-free representations. Thus, we have

Theorem 3.6. *The thick representations of connected simple Lie groups are those on the following list:*

(1) *the trivial* 1*-dimensional representation for any groups*
(2) A_n $(n \geq 1)$
  - *the standard representation V for A_n $(n \geq 1)$ with highest weight ω_1*
  - *the dual representation V^* of V for A_n $(n \geq 1)$ with highest weight ω_n*
  - *the symmetric tensor $S^m(V)$ $(m \geq 2)$ of V for A_1 with highest weight $m\omega_1$*

(3) B_n $(n \geq 2)$
  - *the standard representation V for B_n $(n \geq 2)$ with highest weight ω_1*
  - *the spin representation for B_2 with highest weight ω_2*

(4) C_n $(n \geq 3)$

- *the standard representation V for C_n $(n \geq 3)$ with highest weight ω_1*

(5) G_2
- *the 7-dimensional representation V for G_2 with highest weight ω_1.*

Proof. By Theorem 3.5, it suffices to list up all irreducible representations which are weight multiplicity-free and whose weight posets are totally ordered sets. Using [3, Theorem 4.6.3] and [8, Table 1], we can obtain the list of thick representations of connected simple Lie groups. □

We also have the list of dense representations:

Theorem 3.7. *The dense representations of connected simple Lie groups are those on the following list:*

(1) *the trivial 1-dimensional representation for any groups*

(2) A_n $(n \geq 1)$
- *the standard representation V for A_n $(n \geq 1)$ with highest weight ω_1*
- *the dual representation V^* of V for A_n $(n \geq 1)$ with highest weight ω_n*
- *the symmetric tensor $S^2(V)$ of V for A_1 with highest weight $2\omega_1$*

(3) B_n $(n \geq 2)$
- *the standard representation V for B_n $(n \geq 2)$ with highest weight ω_1.*

Proof. It suffices to verify whether thick representations in the list of Theorems 3.6 are dense or not. It is well-known that the standard representations V of A_n and B_n are dense. We also see that the dual representation V^* of V for A_n is dense. (For A_n, see Example 2.3 or [2, §15.2]. For B_n, see Example 2.5 or [2, Theorem 19.14].) By Corollary 2.10, $S^2(V)$ for A_1 is dense since $\dim S^2(V) = 3$.

Conversely, let us show that $S^m(V)$ for A_1 is not dense if $m \geq 3$. Let $\{\varphi_1 > \varphi_2\}$ be the weight poset of the standard representation V of A_1. The weight poset of $S^m(V)$ is $\{(m-k)\varphi_1 + k\varphi_2 \mid k = 0, 1, 2, \ldots, m\}$. Thereby, the weight poset of $\wedge^2 S^m(V)$ is $\{(2m - k_1 - k_2)\varphi_1 + (k_1 + k_2)\varphi_2 \mid 0 \leq k_1 < k_2 \leq m\}$. If $m \geq 3$, then $\dim \wedge^2 S^m(V)_{(2m-3)\varphi_1 + 3\varphi_2} = 2$ for the cases $(k_1, k_2) = (0, 3), (1, 2)$. Since $\wedge^2 S^m(V)$ is not weight multiplicity-free and any irreducible representations $S^{m'}(V)$ of A_1 are weight multiplicity-free, $\wedge^2 S^m(V)$ is not irreducible. Hence $S^m(V)$ $(m \geq 3)$ is not dense. It is well-known that the first fundamental representations of C_n and G_2 are not dense. (For C_n, see Example 2.6 or [2, §17.2]. For G_2, see [2, §22.3].) The spin representation for B_2 with highest weight ω_2 is not dense since it is equivalent to the first fundamental representation for C_2 (for C_2, see Example 2.6 or [2, §16.2]). Therefore, we obtain the list of dense representations. □

To simplify the classification of thick representations, we introduce the notion of geometric equivalence.

Definition 3.8 (*cf.* [1, §6], [5, §5])**.** For two representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G' \to \mathrm{GL}(V')$, we say that they are *geometrically equivalent* if there exists a $\mathbb{C}$-linear isomorphism $f : V \to V'$ such that for the induced isomorphism $f_* : \mathrm{GL}(V) \to \mathrm{GL}(V')$ we have $f_*(\rho(G)) = \rho'(G')$.

Example 3.9. Let $\rho^* : G \to \mathrm{GL}(V^*)$ be the dual representation of $\rho : G \to \mathrm{GL}(V)$. Then ρ and ρ^* are geometrically equivalent.

Remark 3.10. Assume that two representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G' \to \mathrm{GL}(V')$ are geometric equivalent. Then ρ is thick (resp. dense) if and only if so is ρ'.

According to [4, §3.1], we denote the irreducible representation of a connected simple Lie group G with highest weight ω_1 by G. Similarly, S^mG stands for the m-th symmetric power of G. In addition, let e denote the trivial 1-dimensional representation for any groups G. Then we have:

Theorem 3.11. *If a representation of a connected simple Lie group is thick, then it is geometrically equivalent to one of the following list:*

$$e,\ \mathrm{SL}_n(n \geq 2),\ S^m\mathrm{SL}_2(m \geq 2),\ \mathrm{SO}_{2n+1}(n \geq 2),\ \mathrm{Sp}_{2n}(n \geq 2),\ \mathrm{G}_2.$$

If a representation of a connected simple Lie group is dense, then it is geometrically equivalent to one of the following list:

$$e,\ \mathrm{SL}_n(n \geq 2),\ S^2\mathrm{SL}_2,\ \mathrm{SO}_{2n+1}(n \geq 2).$$

Proof. The last fundamental representation of B_2 with highest weight ω_2 is geometric equivalent to the first fundamental representation of C_2 with highest weight ω_1, that is, Sp_4. By Theorems 3.6 and 3.7, we have the classification above. □

Theorem 3.11 also shows the list of geometrically equivalences of thick (or dense) representations of connected semi-simple Lie groups.

Theorem 3.12. *Any thick representation V of a connected semi-simple Lie group G is geometrically equivalent to one of the list in Theorem 3.11. In particular, the list of geometrically equivalences of thick representations (resp. dense representations) of connected semi-simple Lie groups is the same as that of thick representations (resp. dense representations) of connected simple Lie groups in Theorem 3.11.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a thick representation of a connected semi-simple Lie group G. Take a universal cover $\pi : \widetilde{G} = G_1 \times G_2 \times \cdots \times G_r \to G$, where G_i is a simply-connected simple Lie group for each $i = 1, 2, \ldots, r$. We have a thick representation $\widetilde{\rho} = \rho \circ \pi : \widetilde{G} \to \mathrm{GL}(V)$. Since V is an irreducible representation of $\widetilde{G}$, there exist irreducible representations V_i of G_i $(1 \leq i \leq r)$ such that $V \cong V_1 \otimes V_2 \otimes \cdots \otimes V_r$ as representations of $\widetilde{G}$. By Theorem 3.5, V is WMF as a representation of $\widetilde{G}$ and the weight poset $W_{\widetilde{G}}(V)$ is a totally ordered set. Here, weights in $W_{\widetilde{G}}(V)$ are with respect to a maximal torus $T = T_1 \times T_2 \times \cdots \times T_r$ of $\widetilde{G}$, where T_i is a maximal torus of G_i. The order in $W_{\widetilde{G}}(V)$ is defined with respect to a set $\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_r$ of simple roots of $\widetilde{G}$, where Δ_i is a set of simple roots of G_i. Let $W_{G_i}(V_i)$ be the weight poset (with respect to T_i and Δ_i) of the G_i-module V_i. We can write $W_{\widetilde{G}}(V) = \{\sum_{i=1}^{r} \psi_i \mid \psi_i \in W_{G_i}(V_i)\}$.

Suppose that there exists $1 \leq i < j \leq r$ such that $\widetilde{\rho}(G_i) \neq \{e\}$ and $\widetilde{\rho}(G_j) \neq \{e\}$. Then $\sharp W_{G_i}(V_i) \geq 2$ and $\sharp W_{G_j}(V_j) \geq 2$. Choose $\phi_1, \phi_2 \in W_{G_i}(V_i)$ and $\varphi_1, \varphi_2 \in W_{G_j}(V_j)$ such that $\phi_1 > \phi_2$ and $\varphi_1 > \varphi_2$. Let $\xi = \sum_{k \neq i,j} \psi_k$ be the sum of the highest weights $\psi_k \in W_{G_k}(V_k)$ for $k \neq i, j$. For $\eta_1 = \xi + \phi_1 + \varphi_2, \eta_2 = \xi + \phi_2 + \varphi_1 \in W_{\widetilde{G}}(V)$, neither $\eta_1 > \eta_2$ nor $\eta_1 < \eta_2$ holds. This implies that $W_{\widetilde{G}}(V)$ is not totally ordered, which is a contradiction. Hence, any G_k satisfy $\widetilde{\rho}(G_k) = \{e\}$ except some G_i. Since $V_k = \mathbb{C}$ except for $k = i$, the representation V of $\widetilde{G}$ is geometrically equivalent to the representation V_i of G_i. In particular, the representation V of G is geometrically equivalent to a thick representation V_i of a connected simple Lie group G_i. Therefore, Theorem 3.11 also shows the lists of geometrically equivalences of thick and dense representations of connected semi-simple Lie groups. □

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