THE CLASSIFICATION OF THICK REPRESENTATIONS OF SIMPLE LIE GROUPS

KAZUNORI NAKAMOTO AND YASUHIRO OMODA

Abstract. We characterize finite-dimensional thick representations over $\mathbb{C}$ of connected complex semi-simple Lie groups by irreducible representations which are weight multiplicity-free and whose weight posets are totally ordered sets. Moreover, using this characterization, we give the classification of thick representations over $\mathbb{C}$ of connected complex simple Lie groups.

1. Introduction

In our previous paper [8], we have introduced m-thickness and thickness of group representations. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of a group G. If for any subspaces V_1 and V_2 of V with $\dim V_1 = m$ and $\dim V_2 = \dim V - m$ there exists $g \in G$ such that $(\rho(g)V_1) \oplus V_2 = V$, we say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-thick*. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *thick* if ρ is m-thick for each $0 < m < \dim V$ (Definition 2.1). In [8, Proposition 2.7], we proved that 1-thickness is equivalent to irreducibility (Proposition 2.8). Hence m-thickness is a natural generalization of irreducibility of group representations.

Let G be a connected semi-simple Lie group over $\mathbb{C}$, B a Borel subgroup of G, T a maximal torus which is contained in B. Denote their Lie algebras by $\mathfrak{g}, \mathfrak{b}$ and $\mathfrak{t}$, respectively. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional irreducible representation of G over $\mathbb{C}$. We denote the set of $\mathfrak{t}$-weights in V by $W(V)$. Choosing a set of simple roots for $(\mathfrak{g}, \mathfrak{t})$, we can regard $W(V)$ as a partially ordered set (poset) with respect to the usual root order. This poset $W(V)$ is called the *weight poset*. A representation $\rho : G \to \mathrm{GL}(V)$ is said to be *weight multiplicity-free* if the weight spaces in V are all one-dimensional. We give the following characterization of thickness.

Theorem 1.1 (Theorem 3.5)**.** *An irreducible representation $\rho : G \to \mathrm{GL}(V)$ of a connected semi-simple Lie group G is thick if and only if it is weight multiplicity-free and its weight poset is a totally ordered set.*

Using this characterization, we can classify the complex thick representations of connected semi-simple Lie groups.

2010 *Mathematics Subject Classification.* Primary 22E46; Secondary 22E47, 17B10.
Key words and phrases. thick representation, dense representation, simple Lie group.
The first author was partially supported by JSPS KAKENHI Grant Number JP23540044, JP15K04814, JP20K03509.

Theorem 1.2 (Theorems 3.11 and 3.12)**.** *If a representation of a connected semisimple Lie group is thick, then it is geometrically equivalent to one of the following list:*

$$e,\ \mathrm{SL}_n(n \geq 2),\ S^m\mathrm{SL}_2(m \geq 2),\ \mathrm{SO}_{2n+1}(n \geq 2),\ \mathrm{Sp}_{2n}(n \geq 2),\ \mathrm{G}_2.$$

Here the irreducible representation of a connected simple Lie group G of the highest weight ω_1, where ω_1 is the first fundamental weight, is denoted by G. Similarly, S^mG stands for the m-th symmetric power of G. Let e denote the trivial 1-dimensional representation for any group G. For the definition of geometric equivalence, see Definition 3.8.

We denote by ω_i the i-th fundamental weight for a connected simple Lie group G. In §3, all Lie groups are assumed to be over $\mathbb{C}$ and all representations are finite-dimensional over $\mathbb{C}$.

The authors would like to express their gratitude to the referee for a detailed review and kind suggestions.

2. PRELIMINARIES

A *representation* of a group G on a vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. Then such a map ρ gives V the structure of a G-module. We sometimes call V itself a representation of G and write gv for $\rho(g)(v)$. We recall several definitions and results in our previous paper [8].

Definition 2.1 ([8, Definition 2.1])**.** Let G be a group. Let V be a finite-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-thick* if for any subspaces V_1 and V_2 of V with $\dim V_1 = m$ and $\dim V_2 = \dim V - m$, there exists $g \in G$ such that $(\rho(g)V_1) \oplus V_2 = V$. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *thick* if ρ is m-thick for each $0 < m < \dim V$.

Definition 2.2 ([8, Definition 2.3])**.** Let G be a group. Let V be a finite-dimensional vector space over a field k. We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *m-dense* if the induced representation $\wedge^m \rho : G \to \mathrm{GL}(\bigwedge^m V)$ is irreducible. We also say that a representation $\rho : G \to \mathrm{GL}(V)$ is *dense* if ρ is m-dense for each $0 < m < \dim V$.

We show several examples. See [8] for details.

Example 2.3 (*cf.* [8, Proposition 6.5])**.** Let V be the standard representation of SL_n and V^* the dual representation of V. Then V and V^* are dense.

Example 2.4 ([8, Proposition 6.10])**.** The standard representation of SO_{2n} is m-dense for each $0 < m < 2n$ with $m \neq n$, but not n-thick.

Example 2.5 ([8, Proposition 6.11])**.** The standard representation of SO_{2n+1} is dense.

Example 2.6 ([8, Proposition 6.18])**.** The standard representation of Sp_{2n} is thick, but not m-dense for each $1 < m < 2n-1$.

Let V be a finite-dimensional representation of a group G. For positive integers i and j with $i + j = \dim V$, let us consider the G-equivariant perfect pairing $\bigwedge^i V \otimes \bigwedge^j V \xrightarrow{\wedge} \bigwedge^{\dim V} V \cong k$. For a G-invariant subspace W of $\bigwedge^i V$, put $W^\perp := \{y \in \bigwedge^j V \mid x \wedge y = 0 \text{ for any } x \in W\}$. Then $W^\perp$ is also G-invariant. In particular, $\bigwedge^i V$ is irreducible if and only if so is $\bigwedge^j V$.

Proposition 2.7 ([8, Proposition 2.6])**.** *Let V be an n-dimensional representation of a group G. For each $0 < m < n$, V is m-thick* (*resp. m-dense*) *if and only if V is* $(n-m)$*-thick* (*resp.* $(n-m)$*-dense*)*.*

Proposition 2.8 ([8, Proposition 2.7])**.** *For any finite-dimensional representation V of a group G, the following implications hold for $0 < m < \dim V$:*

$$\begin{array}{ccccc} m\text{-}dense & \Longrightarrow & m\text{-}thick & & \\ & & \Downarrow & & \\ 1\text{-}dense & \Longleftrightarrow & 1\text{-}thick & \Longleftrightarrow & irreducible. \end{array}$$

Corollary 2.9 ([8, Corollary 2.8])**.** *For any finite-dimensional representation of a group G, the following implications hold*:

$$dense \Rightarrow thick \Rightarrow irreducible.$$

Corollary 2.10 ([8, Corollary 2.9])**.** *For any representation V of a group G with* $\dim V \leq 3$*, the following implications hold*:

$$dense \Leftrightarrow thick \Leftrightarrow irreducible.$$

Definition 2.11 ([8, Definition 2.10])**.** Let V be an n-dimensional vector space over a field k. For a d-dimensional subspace V' of V with $0 < d < n$, we can consider a point $[\bigwedge^d V']$ in the projective space $\mathbb{P}(\bigwedge^d V)$. In the sequel, we identify $[\bigwedge^d V']$ with a non-zero vector $\bigwedge^d V' \in \bigwedge^d V$ (which is determined by $[\bigwedge^d V']$ up to scalar) for simplicity. For a vector subspace $W \subset \bigwedge^d V$, we say that W is *realizable* if W contains a non-zero vector $\bigwedge^d V'$ obtained by a d-dimensional subspace V' of V.

We have the following criterion of thickness.

Proposition 2.12 ([8, Proposition 2.11])**.** *Let V be an n-dimensional representation of a group G. For $0 < m < n$, V is not m-thick if and only if there exist G-invariant realizable subspaces $W_1 \subseteq \bigwedge^m V$ and $W_2 \subseteq \bigwedge^{n-m} V$ such that $W_1^\perp = W_2$.*

3. The classification of thick representations of simple Lie groups

Let G be a connected semi-simple Lie group over the complex number field $\mathbb{C}$, B a Borel subgroup of G, T a maximal torus which is contained in B, B^- a Borel subgroup of G opposite to B relative to $T = B \cap B^-$. Denote their Lie algebras by

$\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ and $\mathfrak{b}^-$, respectively. Let V be a finite-dimensional irreducible representation of G over $\mathbb{C}$. We will denote the set of $\mathfrak{t}$-weights in V by $W(V)$. For any weight $\varphi \in W(V)$, let V_φ be the φ-weight space in V. Let Π be the set of simple roots and Δ^+ the set of positive roots for $(\mathfrak{g}, \mathfrak{b})$. We can regard $W(V)$ as a partially ordered set (poset) with respect to the usual root order. More precisely, $\mu > \gamma$ if and only if $\mu - \gamma$ is a nonzero sum of simple roots with nonnegative coefficients. In particular, if $\mu - \gamma$ is a simple root, we say that μ *covers* γ. The partially ordered set $W(V)$ is called the *weight poset*. Following [4, § 4.5], we say that a representation V of G is *weight multiplicity-free* (*WMF*) if the weight spaces in V are all one-dimensional. Howe [4] classified the irreducible representations of connected simple Lie groups which are weight multiplicity-free.

Proposition 3.1. *If a representation V of G is thick, it is weight multiplicity-free.*

Proof. Assume that V is not WMF. Then there exists a weight $\varphi \in W(V)$ such that the dimension of V_φ is larger than one. Let $W^+(\varphi)$ be the set of all weights strictly larger than φ, and $Y^+(\varphi)$ the subspace of V which is spanned by all weight spaces for weights in $W^+(\varphi)$. Because the dimension of V_φ is larger than one, we can choose two linear independent φ-weight vectors v and w. Let $W_\varphi^{a,b}(+)$ be $\mathbb{C}(av + bw) \oplus Y^+(\varphi)$ for $a, b \in \mathbb{C}$. The subspace $W_\varphi^{a,b}(+)$ is B-invariant. Let n be the dimension of V, and d the dimension of $W_\varphi^{a,b}(+)$ for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. The elements $\bigwedge^d W_\varphi^{a,b}(+)$ for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ are distinct B-eigenvectors in $\bigwedge^d V$ with the same weight. Let $U_\varphi^{a,b}(+)$ be the irreducible G-submodule in $\bigwedge^d V$ with the highest weight vector $\bigwedge^d W_\varphi^{a,b}(+)$. Let $U_\varphi(+)$ be the direct sum $U_\varphi^{1,0}(+) \oplus U_\varphi^{0,1}(+) \subset \bigwedge^d V$. Any irreducible G-submodule of $U_\varphi(+)$ is equal to $U_\varphi^{a,b}(+)$ for some $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Hence any irreducible G-submodule of $U_\varphi(+)$ is realizable.

Let $Y^-(\varphi)$ be the subspace of V which is spanned by all weight spaces for weights in $W(V) \setminus \{W^+(\varphi), \varphi\}$. We take a basis $\{v, w, u_1, \ldots, u_s\}$ for V_φ which contains v, w. Let $W_\varphi(-)$ be the subspace of V which is spanned by $\{w, u_1, \ldots, u_s\}$ and $Y^-(\varphi)$. The subspace $W_\varphi(-)$ is invariant under the action of the opposite Borel subgroup B^-. The equalities $\dim W_\varphi(-) = \dim V - \dim W_\varphi^{a,b}(+) = n - d$ hold for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then $\bigwedge^{n-d} W_\varphi(-)$ is a B^--eigenvector in $\bigwedge^{n-d} V$. Let $U_\varphi(-)$ be the irreducible G-submodule with the lowest weight vector $\bigwedge^{n-d} W_\varphi(-)$ in $\bigwedge^{n-d} V$. Obviously, $U_\varphi(-)$ is realizable. The irreducibility of $U_\varphi(-)$ shows the irreducibility of $\bigwedge^d V/(U_\varphi(-))^\perp$. Then $(U_\varphi(-))^\perp \cap U_\varphi(+) \neq \{0\}$ because $U_\varphi(+)$ is not irreducible. Hence $(U_\varphi(-))^\perp \cap U_\varphi(+)$ contains some realizable G-submodule $U_\varphi^{a,b}(+)$. Therefore $(U_\varphi(-))^\perp$ is realizable. Putting $W_1 = (U_\varphi(-))^\perp$ and $W_2 = U_\varphi(-)$, we see that V is not thick by Proposition 2.12. Hence if V is thick, then it is WMF. □

Proposition 3.2. *If a representation V of G is thick, its weight poset $W(V)$ is a totally ordered set.*

Proof. Let V be a thick representation of G. By Proposition 3.1, V is WMF. For any weight $\phi \in W(V)$, let $W^+(\phi)$ be the set of all weights strictly larger than ϕ, and $Y^+(\phi)$ the subspace of V which is spanned by all weight spaces for weights in $W^+(\phi)$. Note that the irreducible representation V has a highest weight ω and that each weight of V has the form $\omega - \sum_{i=1}^{l} m_i\alpha_i$ ($m_i \in \mathbb{N}$), where $\Pi = \{\alpha_1, \ldots, \alpha_l\}$.

Suppose that the weight poset $W(V)$ is not a totally ordered set. There exists an integer $d > 1$ such that $W(V)$ has the $(d-1)$-st highest weight, but not the d-th highest weight. Let φ be the $(d-1)$-st highest weight, and ψ_1, ψ_2 maximal weights in $W(V) \setminus (W^+(\varphi) \cup \{\varphi\})$. Then the subset $W^+(\varphi) \cup \{\varphi\}$ is a totally ordered set, φ covers ψ_1, ψ_2, and $W^+(\psi_1) = W^+(\psi_2) = W^+(\varphi) \cup \{\varphi\}$. Because V is WMF, there exists a unique ψ_i-weight vector v_i up to scalar for each $i = 1, 2$. Let $W_{\psi_i}(+)$ be $\mathbb{C}v_i \oplus Y^+(\psi_i)$. The subspaces $W_{\psi_i}(+)$ are B-invariant for each $i = 1, 2$. Let n be the dimension of V. Note that $\dim W_{\psi_i}(+) = d$ for $i = 1, 2$. The elements $\bigwedge^d W_{\psi_1}(+)$ and $\bigwedge^d W_{\psi_2}(+)$ are distinct B-eigenvectors with distinct weights in $\bigwedge^d V$. Let $U_{\psi_i}(+)$ be the irreducible G-submodule of $\bigwedge^d V$ with the highest weight vector $\bigwedge^d W_{\psi_i}(+)$ for each $i = 1, 2$. Then $U_{\psi_1}(+)$ and $U_{\psi_2}(+)$ are realizable and not isomorphic to each other as G-modules. Let $Y^-(\psi_1)$ be the subspace of V which is spanned by all weight spaces for weights in $W(V) \setminus (W^+(\psi_1) \cup \{\psi_1\})$. The subspace $Y^-(\psi_1)$ is invariant under the action of the opposite Borel subgroup B^-. The equalities $\dim Y^-(\psi_1) = \dim V - \dim W_{\psi_1}(+) = n - d$ hold. Then $\bigwedge^{n-d} Y^-(\psi_1)$ is a B^--eigenvector in $\bigwedge^{n-d} V$. Let $U_{\psi_1}(-)$ be the irreducible G-submodule of $\bigwedge^{n-d} V$ with the lowest weight vector $\bigwedge^{n-d} Y^-(\psi_1)$. Then $U_{\psi_1}(-)$ is realizable. The irreducibility of $U_{\psi_1}(-)$ shows the irreducibility of $\bigwedge^d V/(U_{\psi_1}(-))^\perp$. Then $(U_{\psi_1}(-))^\perp \cap (U_{\psi_1}(+) \oplus U_{\psi_2}(+)) \neq \{0\}$. Because $U_{\psi_1}(+)$ is not isomorphic to $U_{\psi_2}(+)$, $U_{\psi_1}(+) \subset (U_{\psi_1}(-))^\perp$ or $U_{\psi_2}(+) \subset (U_{\psi_1}(-))^\perp$. In particular, $(U_{\psi_1}(-))^\perp$ is realizable. Putting $W_1 = (U_{\psi_1}(-))^\perp$ and $W_2 = U_{\psi_1}(-)$, we see that V is not thick by Proposition 2.12. This is a contradiction. Hence $W(V)$ is a totally ordered set. $\square$

Let us denote the Grassmann variety which is the set of all k-dimensional subspaces of a vector space V by $\mathrm{Grass}(k, V) (\subset \mathbb{P}(\bigwedge^k V))$.

Lemma 3.3. *Let V be a representation of G, and W a G-invariant realizable subspace of $\bigwedge^k V$. Then there exists $[v] \in \mathbb{P}(W) \cap \mathrm{Grass}(k, V)$ such that $[v]$ is B-invariant.*

Proof. Let X be $\mathbb{P}(W) \cap \mathrm{Grass}(k, V)$. Because W is realizable, X is not empty. Note that X is G-invariant and compact. We take a G-orbit O in X whose dimension is minimal. The orbit O is closed and then compact. There is a parabolic subgroup P of G such that the orbit O is isomorphic to G/P. Then there is a point $[v] \in O \subset \mathbb{P}(W) \cap \mathrm{Grass}(k, V)$ such that $[v]$ is B-invariant. $\square$

Lemma 3.4. *Assume that an irreducible representation V of G is weight multiplicity-free, its weight poset $W(V)$ is a totally ordered set $\{\varphi_1 > \varphi_2 > \cdots > \varphi_n\}$, and*

W is a G-invariant realizable subspace of $\bigwedge^k V$. Let v_i be a nonzero vector in the φ_i-weight space V_{φ_i} $(i = 1, 2, \ldots, n)$. Then W contains $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ and $v_{n-(k-1)} \wedge v_{n-(k-2)} \wedge \cdots \wedge v_n$.

Proof. Because V is weight multiplicity-free, $\{v_1, \ldots, v_n\}$ is a basis of V. By Lemma 3.3, there exists $[v] \in \mathbb{P}(W) \cap \mathrm{Grass}(k, V)$ such that v is a highest weight vector of an irreducible subrepresentation of W with respect to B. We can put

$$\begin{aligned} v = \quad & (p_{1,1}v_1 + p_{1,2}v_2 + \cdots + p_{1,n}v_n) \\ & \wedge(p_{2,1}v_1 + p_{2,2}v_2 + \cdots + p_{2,n}v_n) \\ & \qquad \vdots \\ & \wedge(p_{k,1}v_1 + p_{k,2}v_2 + \cdots + p_{k,n}v_n) \end{aligned}$$

up to scalar multiplication, where $P = (p_{i,j})$ is in reduced row echelon form. Remark that P is uniquely determined. Let X_α be a root vector for a positive root $\alpha \in \Delta^+$. Then $X_\alpha v = 0$ holds for any $\alpha \in \Delta^+$. If $p_{1,1} = p_{1,2} = \cdots = p_{1,i} = 0$ and $p_{1,i+1} = 1$ for $i \geq 1$, there is a positive root $\alpha \in \Delta^+$ such that $X_\alpha v_{i+1}$ is cv_i for a nonzero constant c. Then $X_\alpha v$ is not 0. This is a contradiction. So $p_{1,1} = 1$. Similarly, we can show that $p_{22} = \cdots = p_{kk} = 1$. Because v is a highest weight vector, for any $t \in \mathfrak{t}$ there is a constant c such that $tv = cv$. Then by the uniqueness of P we can show that $p_{ij} = 0$ for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$. Then $v = v_1 \wedge v_2 \wedge \cdots \wedge v_k \in W$. A similar argument with respect to B^- shows that $v_{n-(k-1)} \wedge v_{n-(k-2)} \wedge \cdots \wedge v_n \in W$. □

Theorem 3.5. *An irreducible representation V of a connected semi-simple Lie group G is thick if and only if it is weight multiplicity-free and its weight poset is a totally ordered set.*

Proof. The "only if" part can be proved by Propositions 3.1 and 3.2. Let us prove the "if" part. Let us use the notations in Lemma 3.4. Assume that $W_1 \subseteq \bigwedge^k V$ and $W_2 \subseteq \bigwedge^{n-k} V$ are G-invariant realizable subspaces. By Lemma 3.4, $v_1 \wedge v_2 \wedge \cdots \wedge v_k \in W_1$ and $v_{k+1} \wedge v_{k+2} \wedge \cdots \wedge v_n \in W_2$. Since $(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \wedge (v_{k+1} \wedge v_{k+2} \wedge \cdots \wedge v_n) \neq 0$, $W_1^\perp \neq W_2$. By Proposition 2.12, V is thick. □

By [4, Theorem 4.6.3], we have Howe's classification of irreducible representations of connected simple Lie groups which are weight multiplicity-free. We also refer to Panyushev's paper [9, Table 1] for the weight posets of weight multiplicity-free representations. Thus, we have

Theorem 3.6. *The thick representations of connected simple Lie groups are those on the following list:*

1. *the trivial 1-dimensional representation for any groups*
2. A_n $(n \geq 1)$
    - *the standard representation V for A_n $(n \geq 1)$ with highest weight ω_1*
    - *the dual representation V^* of V for A_n $(n \geq 1)$ with highest weight ω_n*

   - *the symmetric tensor* $S^m(V)$ $(m \geq 2)$ *of* V *for* A_1 *with highest weight* $m\omega_1$
(3) B_n $(n \geq 2)$
   - *the standard representation* V *for* B_n $(n \geq 2)$ *with highest weight* ω_1
   - *the spin representation for* B_2 *with highest weight* ω_2

(4) C_n $(n \geq 3)$
   - *the standard representation* V *for* C_n $(n \geq 3)$ *with highest weight* ω_1

(5) G_2
   - *the* 7*-dimensional representation* V *for* G_2 *with highest weight* ω_1*.*

Proof. By Theorem 3.5, it suffices to list up all irreducible representations which are weight multiplicity-free and whose weight posets are totally ordered sets. Using [4, Theorem 4.6.3] and [9, Table 1], we can obtain the list of thick representations of connected simple Lie groups. □

We also have the list of dense representations:

Theorem 3.7. *The dense representations of connected simple Lie groups are those on the following list:*

(1) *the trivial* 1*-dimensional representation for any groups*
(2) A_n $(n \geq 1)$
   - *the standard representation* V *for* A_n $(n \geq 1)$ *with highest weight* ω_1
   - *the dual representation* V^* *of* V *for* A_n $(n \geq 1)$ *with highest weight* ω_n
   - *the symmetric tensor* $S^2(V)$ *of* V *for* A_1 *with highest weight* $2\omega_1$

(3) B_n $(n \geq 2)$
   - *the standard representation* V *for* B_n $(n \geq 2)$ *with highest weight* ω_1*.*

Proof. It suffices to verify whether thick representations in the list of Theorems 3.6 are dense or not. It is well-known that the standard representations V of A_n and B_n are dense. We also see that the dual representation V^* of V for A_n is dense. (For A_n, see Example 2.3 or [3, §15.2]. For B_n, see Example 2.5 or [3, Theorem 19.14].) By Corollary 2.10, $S^2(V)$ for A_1 is dense since $\dim S^2(V) = 3$.

Conversely, let us show that $S^m(V)$ for A_1 is not dense if $m \geq 3$. Let $\{\varphi_1 > \varphi_2\}$ be the weight poset of the standard representation V of A_1. The weight poset of $S^m(V)$ is $\{(m-k)\varphi_1 + k\varphi_2 \mid k = 0, 1, 2, \ldots, m\}$. Thereby, the weight poset of $\bigwedge^2 S^m(V)$ is $\{(2m - k_1 - k_2)\varphi_1 + (k_1 + k_2)\varphi_2 \mid 0 \leq k_1 < k_2 \leq m\}$. If $m \geq 3$, then $\dim \bigwedge^2 S^m(V)_{(2m-3)\varphi_1 + 3\varphi_2} = 2$ for the cases $(k_1, k_2) = (0, 3), (1, 2)$. Since $\bigwedge^2 S^m(V)$ is not weight multiplicity-free and any irreducible representations $S^{m'}(V)$ of A_1 are weight multiplicity-free, $\bigwedge^2 S^m(V)$ is not irreducible. Hence $S^m(V)$ $(m \geq 3)$ is not dense. It is well-known that the first fundamental representations of C_n and G_2 are not dense. (For C_n, see Example 2.6 or [3, §17.2]. For G_2, see [3, §22.3].) The spin representation for B_2 with highest weight ω_2 is not dense since it is equivalent to the first fundamental representation for C_2 (for C_2, see Example 2.6 or [3, §16.2]). Therefore, we obtain the list of dense representations. □

According to [1, §6], [7, §5], and so on, we introduce the notion of geometric equivalence for simplifying the classification of thick representations.

Definition 3.8 (*cf.* [1, §6], [7, §5])**.** For two representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G' \to \mathrm{GL}(V')$, we say that they are *geometrically equivalent* if there exists a $\mathbb{C}$-linear isomorphism $f : V \to V'$ such that $\rho'(G') = f\rho(G)f^{-1}$.

We prove the following proposition which was known in [7, §5].

Proposition 3.9 ([7, §5])**.** *Let G be a connected semi-simple Lie group over $\mathbb{C}$. Let $\rho^* : G \to \mathrm{GL}(V^*)$ be the dual representation of a finite-dimensional irreducible representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{C}$. Then ρ and ρ^* are geometrically equivalent.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. Fix a set of simple roots Π of the root system Δ. Let λ be the highest weight of V with respect to Π and w_0 the longest element of the Weyl group W. Then $-w_0(\lambda)$ is the highest weight of V^* (see [5, Excercises 10.9 and 21.6]). Let $\phi' : \mathfrak{h} \to \mathfrak{h}$ be the isomorphism whose dual $\phi'^* : \mathfrak{h}^* \to \mathfrak{h}^*$ is given by $\mu \mapsto -w_0(\mu)$. There exists a Lie algebra isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ extending ϕ' (see [5, Theorem 18.4 (b)]). Take a universal cover $\pi : \widetilde{G} \to G$. The dual representation $\widetilde{\rho}^*$ of $\widetilde{\rho} = \rho \circ \pi : \widetilde{G} \to \mathrm{GL}(V)$ can be identified with $\rho^* \circ \pi$. By [2, Chapter III, §6, Theorem 1], there exists an automorphism $\psi : \widetilde{G} \to \widetilde{G}$ such that $d\psi = \phi$. Since $\widetilde{\rho} \circ \psi$ and $\widetilde{\rho}^*$ have the same highest weight $-w_0(\lambda)$, there exists an isomorphism $f : V \to V^*$ such that $(\rho^* \circ \pi)(\widetilde{g}) = \widetilde{\rho}^*(\widetilde{g}) = f \circ (\widetilde{\rho} \circ \psi)(\widetilde{g}) \circ f^{-1}$ for any $\widetilde{g} \in \widetilde{G}$. Hence $\rho^*(G) = \widetilde{\rho}^*(\widetilde{G}) = f((\widetilde{\rho} \circ \psi)(\widetilde{G}))f^{-1} = f\rho(G)f^{-1}$. Therefore ρ and ρ^* are geometrically equivalent. □

Remark 3.10. Assume that two representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G' \to \mathrm{GL}(V')$ are geometrically equivalent. Then ρ is thick (resp. dense) if and only if so is ρ'.

According to [6, §3.1], we denote the irreducible representation of a connected simple Lie group G with highest weight ω_1 by G. Similarly, S^mG stands for the m-th symmetric power of G. In addition, let e denote the trivial 1-dimensional representation for any groups G. Then we have:

Theorem 3.11. *If a representation of a connected simple Lie group is thick, then it is geometrically equivalent to one of the following list:*

$$e,\ \mathrm{SL}_n (n \geq 2),\ S^m\mathrm{SL}_2 (m \geq 2),\ \mathrm{SO}_{2n+1} (n \geq 2),\ \mathrm{Sp}_{2n} (n \geq 2),\ \mathrm{G}_2.$$

If a representation of a connected simple Lie group is dense, then it is geometrically equivalent to one of the following list:

$$e,\ \mathrm{SL}_n (n \geq 2),\ S^2\mathrm{SL}_2,\ \mathrm{SO}_{2n+1} (n \geq 2).$$

Proof. The last fundamental representation of B_2 with highest weight ω_2 is geometrically equivalent to the first fundamental representation of C_2 with highest weight ω_1, that is, Sp_4. By Theorems 3.6 and 3.7, we have the classification above. □

Theorem 3.11 also shows the list of geometric equivalence classes of thick (or dense) representations of connected semi-simple Lie groups.

Theorem 3.12. *Any thick representation V of a connected semi-simple Lie group G is geometrically equivalent to one of the list in Theorem 3.11. In particular, the list of geometric equivalence classes of thick representations (resp. dense representations) of connected semi-simple Lie groups is the same as that of thick representations (resp. dense representations) of connected simple Lie groups in Theorem 3.11.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a thick representation of a connected semi-simple Lie group G. Take a universal cover $\pi : \widetilde{G} = G_1 \times G_2 \times \cdots \times G_r \to G$, where G_i is a simply-connected simple Lie group for each $i = 1, 2, \ldots, r$. We have a thick representation $\widetilde{\rho} = \rho \circ \pi : \widetilde{G} \to \mathrm{GL}(V)$. Since V is an irreducible representation of $\widetilde{G}$, there exist irreducible representations V_i of G_i $(1 \leq i \leq r)$ such that $V \cong V_1 \otimes V_2 \otimes \cdots \otimes V_r$ as representations of $\widetilde{G}$. By Theorem 3.5, V is WMF as a representation of $\widetilde{G}$ and the weight poset $W_{\widetilde{G}}(V)$ is a totally ordered set. Here, weights in $W_{\widetilde{G}}(V)$ are with respect to a maximal torus $T = T_1 \times T_2 \times \cdots \times T_r$ of $\widetilde{G}$, where T_i is a maximal torus of G_i. The order in $W_{\widetilde{G}}(V)$ is defined with respect to a set $\Pi = \Pi_1 \sqcup \Pi_2 \sqcup \cdots \sqcup \Pi_r$ of simple roots of $\widetilde{G}$, where Π_i is a set of simple roots of G_i. Let $W_{G_i}(V_i)$ be the weight poset (with respect to T_i and Π_i) of the G_i-module V_i. We can write $W_{\widetilde{G}}(V) = \{\sum_{i=1}^{r} \psi_i \mid \psi_i \in W_{G_i}(V_i)\}$.

Suppose that there exists $1 \leq i < j \leq r$ such that $\widetilde{\rho}(G_i) \neq \{e\}$ and $\widetilde{\rho}(G_j) \neq \{e\}$. Then $\sharp W_{G_i}(V_i) \geq 2$ and $\sharp W_{G_j}(V_j) \geq 2$. Choose $\phi_1, \phi_2 \in W_{G_i}(V_i)$ and $\varphi_1, \varphi_2 \in W_{G_j}(V_j)$ such that $\phi_1 > \phi_2$ and $\varphi_1 > \varphi_2$. Let $\xi = \sum_{k \neq i,j} \psi_k$ be the sum of the highest weights $\psi_k \in W_{G_k}(V_k)$ for $k \neq i, j$. For $\eta_1 = \xi + \phi_1 + \varphi_2, \eta_2 = \xi + \phi_2 + \varphi_1 \in W_{\widetilde{G}}(V)$, neither $\eta_1 > \eta_2$ nor $\eta_1 < \eta_2$ holds. This implies that $W_{\widetilde{G}}(V)$ is not totally ordered, which is a contradiction. Hence, any G_k satisfy $\widetilde{\rho}(G_k) = \{e\}$ except some G_i. Since $V_k = \mathbb{C}$ except for $k = i$, the representation V of $\widetilde{G}$ is geometrically equivalent to the representation V_i of G_i. In particular, the representation V of G is geometrically equivalent to a thick representation V_i of a connected simple Lie group G_i. Therefore, Theorem 3.11 also shows the lists of geometric equivalence classes of thick and dense representations of connected semi-simple Lie groups. □

References

[1] C. Benson and G. Ratcliff, On multiplicity free actions, Representations of real and p-adic groups, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., **2**, Singapore Univ. Press, Singapore, 2004, 221–304.

[2] N. Bourbaki, Lie groups and Lie algebras, Chapters 1—3, Translated from the French, Reprint of the 1989 English translation, Springer-Verlag, Berlin, 1998.

[3] W. Fulton and J. Harris, Representation theory—A first course, Graduate Texts in Mathematics, **129**, Springer-Verlag, New York, 1991.
[4] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proc., **8**, Bar-Ilan Univ., Ramat Gan, 1995, 1–182.
[5] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics, **9**, Springer-Verlag, New York-Berlin, 1972.
[6] V. G. Kac, Some remarks on nilpotent orbits, J. Algebra **64** (1980), 190–213.
[7] F. Knop, Some remarks on multiplicity free spaces, Representation theories and algebraic geometry, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., **514**, Kluwer Acad. Publ., Dordrecht, 1998, 301–317.
[8] K. Nakamoto and Y. Omoda, Thick representations and dense representations I, Kodai Math. J. **42** (2019), 274–307.
[9] D. I. Panyushev, Properties of weight posets for weight multiplicity free representations, Mosc. Math. J. **9** (2009), 867–883, 935–936.

Center for Medical Education and Sciences, Faculty of Medicine, University of Yamanashi
Email address: nakamoto@yamanashi.ac.jp

Natural Sciences Division, National Institute of Technology, Akashi College
Email address: omoda@akashi.ac.jp