HOCHSCHILD COHOMOLOGY OF THE QUADRATIC MONOMIAL ALGEBRA N_m

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Abstract. Let $\mathrm{N}_m(R) = \{(a_{ij}) \in \mathrm{M}_m(R) \mid a_{11} = a_{22} = \cdots = a_{mm} \text{ and } a_{ij} = 0 \text{ for any } i > j\}$ for a commutative ring R. Then $\mathrm{N}_m(R)$ is a quadratic monomial algebra over R. We calculate $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ as R-modules. We also determine the R-algebra structure of the Hochschild cohomology ring $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$. For $m \geq 3$, $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is an infinitely generated algebra over R and has no Batalin-Vilkovisky algebra structure giving the Gerstenhaber bracket.

1. Introduction

For a commutative ring R, set

$$\mathrm{N}_m(R) = \left\{ \begin{pmatrix} a & * & * & \cdots & * \\ 0 & a & * & \cdots & * \\ 0 & 0 & a & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in \mathrm{M}_m(R) \right\}.$$

In this paper, we calculate $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ as R-modules. We also calculate the Hochschild cohomology ring $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ as R-algebras. Moreover, we calculate the Gerstenhaber bracket on $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ and show that $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ has no Batalin-Vilkovisky algebra structure which gives the Gerstenhaber bracket.

In [9], we have calculated the Hochschild cohomology $\mathrm{HH}^*(A, \mathrm{M}_3(k)/A)$ for any k-subalgebra A of $\mathrm{M}_3(k)$ over an algebraically closed field k. The k-subalgebra $\mathrm{N}_3(k)$ is one of the most difficult k-subalgebras of $\mathrm{M}_3(k)$ to calculate $\mathrm{HH}^*(A, \mathrm{M}_3(k)/A)$. Indeed, we could not calculate $\mathrm{HH}^*(\mathrm{N}_3(k), \mathrm{M}_3(k)/\mathrm{N}_3(k))$ until we used spectral sequences. Hence, it is a challenging task to calculate $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ for $m \geq 4$. It is also a hard job to determine the R-algebra structure of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ for $m \geq 3$.

Setting $x_1 = E_{1,2}, x_2 = E_{2,3}, \ldots, x_{m-1} = E_{m-1,m} \in \mathrm{N}_m(R)$, we have an isomorphism as R-algebras

$$\mathrm{N}_m(R) \cong R\langle x_1, x_2, \ldots, x_{m-1}\rangle / \langle x_i x_j \mid j \neq i+1\rangle,$$

where $E_{i,j}$ is the (i,j)-th matrix unit in $\mathrm{M}_m(R)$. Note that $\mathrm{N}_m(R)$ is a quadratic monomial algebra over R with degree $|x_i| = 1$ $(1 \leq i \leq m-1)$. When R is a field, $\mathrm{N}_m(R)$ is a Koszul algebra over R.

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The quadratic dual algebra $\mathrm{N}_m(R)^!$ of $\mathrm{N}_m(R)$ is isomorphic to $R\langle y_1, y_2, \ldots, y_{m-1}\rangle/\langle y_i y_{i+1} \mid 1 \leq i \leq m-2\rangle$. Then $\mathrm{N}_m(R)^!$ is also a graded R-algebra with degree $|y_i| = 1$ $(1 \leq i \leq m-1)$. Put

$$\varphi(d) = \mathrm{rank}_R \mathrm{N}_m(R)^!_d,$$

where $\mathrm{N}_m(R)^!_d$ is the homogeneous part of $\mathrm{N}_m(R)^!$ of degree d. The Poincaré series $f^!(t) = \sum_{d\geq 0} \varphi(d)t^d$ of $\mathrm{N}_m(R)^!$ can be calculated by $f^!(t) = 1/(1 + \sum_{k=1}^{m-1}(-1)^k(m-k)t^k)$ (Proposition 4.15).

The first main theorem is the following:

Theorem 1.1 (Theorem 5.11 and Corollary 5.13)**.** *Let $m \geq 3$. The Hochschild cohomology $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ is a free R-module for $n \geq 0$. The rank of $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ for $n \geq 0$ is given by*

$$\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) = \begin{cases} m-1 & (n = 0), \\ (m-2)\varphi(n) & (n > 0). \end{cases}$$

The second main theorems are the following:

Theorem 1.2 (Theorem 6.34)**.** *Let $m \geq 3$. The Hochschild cohomology $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is a free R-module for $n \geq 0$. The rank of $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is given by*

$$\begin{aligned} &\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R)) \\ = &\begin{cases} 2 & (n = 0), \\ 2m-4 & (n = 1), \\ \varphi(n) + (m-4)\varphi(n-1) + (-1)^m \varphi(n-m+1) + \sum_{k=2}^{m-1}(-1)^k(k+1)\varphi(n-k) & (n \geq 2). \end{cases} \end{aligned}$$

Theorem 1.3 (Theorems 7.2 and 7.4 and Corollary 7.5)**.** *Let $m \geq 3$. There is an augmentation map $\epsilon : \mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R)) \to R$ as an R-algebra homomorphism such that the Kernel $\overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ of ϵ satisfies*

$$\overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R)) \cdot \overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R)) = 0.$$

In particular, $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is an infinitely generated algebra over R.

Theorem 1.4 (Theorem 8.18)**.** *For $m \geq 3$, $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ has no Batalin-Vilkovisky algebra structure over R which gives the Gerstenhaber bracket $[\ ,\ ]$.*

As an application of the main theorems, we can calculate the dimension of the tangent space of the moduli of subalgebras of M_m over $\mathbb{Z}$ at N_m. Set $d = \mathrm{rank}_R \mathrm{N}_m(R) = \dfrac{m^2 - m + 2}{2}$.

Theorem 1.5 (Theorem 5.18)**.** *The dimension of the Zariski tangent space of the moduli of rank d subalgebras of M_m over $\mathbb{Z}$ at N_m is*

$$\dim T_{\mathrm{Mold}_{m,d}/\mathbb{Z}, \mathrm{N}_m} = \frac{3m^2 - 7m + 4}{2}$$

for $m \geq 3$.

The organization of this paper is as follows: in Section 2, we review Hochschild cohomology. In Section 3, we introduce several results on spectral sequences. In Section 4, we show that $\mathrm{HH}^*(\mathrm{N}_m(R), R) \cong \mathrm{N}_m(R)^!$ as R-algebras. We also describe the Poincaré series $f^!(t)$ of $\mathrm{N}_m(R)^!$ explicitly. In Section 5, we calculate $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ as R-modules. We also calculate the dimension of the tangent space of the moduli of subalgebras of M_m over $\mathbb{Z}$ at N_m. In Section 6, we determine the R-module structure of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$. In Section 7, we determine the product structure of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$. In Section 8, we describe the Gerstenhaber bracket on $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$. We also show that $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ has no Batalin-Vilkovisky algebra structure giving the Gerstenhaber bracket $[\ ,\ ]$ for $m \geq 3$. In Section 9, we deal with $\mathrm{HH}^*(\mathrm{N}_2(R), \mathrm{M}_2(R)/\mathrm{N}_2(R))$ and $\mathrm{HH}^*(\mathrm{N}_2(R), \mathrm{N}_2(R))$ as an appendix.

Throughout this paper, R denotes a commutative ring and A an associative algebra over R. We denote by $E_{i,j} \in \mathrm{M}_m(R)$ the matrix with entry 1 in the (i,j)-component and 0 the other components. We also denote by I_m the identity matrix in $\mathrm{M}_m(R)$. By a *module* M over A, we mean a left module M over A, unless stated otherwise. We set $\mathrm{B}_m(R) = \{(a_{ij}) \in \mathrm{M}_m(R) \mid a_{ij} = 0 \text{ for } i > j\}$ and $J(\mathrm{N}_m(R)) = \{(a_{ij}) \in \mathrm{M}_m(R) \mid a_{ij} = 0 \text{ for } i \geq j\}$. For a subset S of an R-module M, we denote by $R\{S\}$ the R-submodule of M generated by S. For a homogeneous element x of a graded R-module $M = \oplus_{i\in\mathbb{Z}} M_i$ (or a graded R-algebra $A = \oplus_{i\in\mathbb{Z}} A_i$), we denote by $|x|$ the degree of x.

2. Preliminaries on Hochschild cohomology

In this section, we make a brief survey of Hochschild cohomology (cf. [3] and [14]). Throughout this section, M denotes an A-bimodule over R.

Definition 2.1. Assume that A is a projective module over R. Let $A^e = A \otimes_R A^{op}$ be the enveloping algebra of A. For A-bimodules A and M over R, we can regard them as A^e-modules. We define the i-th Hochschild cohomology group $\mathrm{HH}^i(A, M)$ as $\mathrm{Ext}^i_{A^e}(A, M)$.

We denote by $B_*(A, A, A)$ the bar resolution of A as A-bimodules over R. For $p \geq 0$, we have

$$B_p(A, A, A) = A \otimes_R \overbrace{A \otimes_R \cdots \otimes_R A}^{p} \otimes_R A.$$

For an A-bimodule M over R, we define a cochain complex $C^*(A, M)$ to be

$$\mathrm{Hom}_{A^e}(B_*(A, A, A), M).$$

We can identify $C^p(A, M)$ with an R-module

$$\mathrm{Hom}_R(\overbrace{A \otimes_R \cdots \otimes_R A}^{p}, M).$$

Under this identification, the coboundary map $d^p : C^p(A, M) \to C^{p+1}(A, M)$ is given by

$$\begin{aligned} d^p(f)(a_1 \otimes \cdots \otimes a_{p+1}) &= a_1 \cdot f(a_2 \otimes \cdots \otimes a_{p+1}) \\ &\quad + \sum_{i=1}^{p} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1}) \\ &\quad + (-1)^{p+1} f(a_1 \otimes \cdots \otimes a_p) \cdot a_{p+1} \end{aligned}$$

for $f \in C^p(A, M)$ $(p \geq 1)$ and

$$d^0(m)(a) = am - ma$$

for $m \in C^0(A, M) = M$. The Hochschild cohomology group $\mathrm{HH}^*(A, M)$ of A with coefficients in M can be calculated by taking the cohomology of the cochain complex $C^*(A, M)$:

$$\mathrm{HH}^*(A, M) = H^*(C^*(A, M)).$$

Remark 2.2. In Definition 2.1, the assumption that A is a projective module over R is needed for $\mathrm{Ext}^i_{A^e}(A, M) \cong H^i(C^*(A, M))$ for $i \geq 0$.

Let N be another A-bimodule over R. We define a map

$$\cup : C^*(A,M) \times C^*(A,N) \longrightarrow C^*(A, M \otimes_A N)$$

by

$$(f \cup g)(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q) = f(a_1 \otimes \cdots \otimes a_p) \otimes g(b_1 \otimes \cdots \otimes b_q)$$

for $f \in C^p(A,M)$ and $g \in C^q(A,N)$. The map $\cup$ is R-bilinear and satisfies

$$d^{p+q}(f \cup g) = d^p(f) \cup g + (-1)^p f \cup d^q(g).$$

Hence the map $\cup$ induces a map

$$\mathrm{HH}^p(A,M) \otimes_R \mathrm{HH}^q(A,N) \longrightarrow \mathrm{HH}^{p+q}(A, M \otimes_A N)$$

of R-modules.

By the above construction, we see that the Hochschild cohomology $\mathrm{HH}^*(A,-)$ defines a lax monoidal functor from the monoidal category of A-bimodules over R to the monoidal category of graded R-modules. Hence, $\mathrm{HH}^*(A,M)$ is a graded associative algebra over R if M is a monoid object in the category of A-bimodules over R.

Suppose that the unit map $R \to A$ is a split monomorphism. We set $\overline{A} = A/RI$, where $I \in A$ is the image of $1 \in R$ under the unit map. Let $\overline{B}_*(A,A,A)$ be the reduced bar resolution of A as A-bimodules over R. We have

$$\overline{B}_p(A,A,A) \cong A \otimes_R \overbrace{\overline{A} \otimes_R \cdots \otimes_R \overline{A}}^{p} \otimes_R A$$

for $p \geq 0$. For an A-bimodule M over R, we denote the cochain complex $\mathrm{Hom}_{A^e}(\overline{B}_*(A,A,A), M)$ by $\overline{C}^*(A,M)$. The cochain complex $\overline{C}^*(A,M)$ is a subcomplex of $C^*(A,M)$. Recall that the reduced bar resolution $\overline{B}_*(A,A,A)$ is chain homotopy equivalent to the bar resolution $B_*(A,A,A)$. Hence, the inclusion $\overline{C}^*(A,M) \to C^*(A,M)$ induces an isomorphism

$$H^*(\overline{C}^*(A,M)) \cong \mathrm{HH}^*(A,M).$$

We observe that the map $\cup : C^*(A,M) \times C^*(A,N) \to C^*(A, M \otimes_A N)$ induces an R-bilinear map

$$\cup : \overline{C}^*(A,M) \times \overline{C}^*(A,N) \longrightarrow \overline{C}^*(A, M \otimes_A N),$$

where N is another A-bimodule over R. Hence the map $\cup : \overline{C}^*(A,M) \times \overline{C}^*(A,N) \to \overline{C}^*(A, M \otimes_A N)$ induces the same map $\mathrm{HH}^p(A,M) \otimes_R \mathrm{HH}^q(A,N) \to \mathrm{HH}^{p+q}(A, M \otimes_A N)$ of R-modules as before.

3. Spectral sequences

In this section we recall the construction of spectral sequences associated to filtered cochain complexes. In particular, we construct a spectral sequence by introducing a filtration on the Hochschild cochain complex $C^*(A,M)$ by powers of a two-sided ideal (e.g. the Jacobson radical J) of A.

3.1. **Review on the construction of spectral sequences associated to filtered cochain complexes.** We can consider spectral sequences in an abelian category $\mathcal{A}$. In this subsection we recall the construction of spectral sequences associated to filtered cochain complexes in $\mathcal{A}$.

Let (C^*, d) be a cochain complex in $\mathcal{A}$ equipped with a filtration

$$C^* = F^0C^* \supset F^1C^* \supset \cdots \supset F^pC^* \supset \cdots$$

by subcomplexes. Throughout this paper we assume that there exists $t \in \mathbb{Z}_{>0}$ such that $F^tC^* = 0$. We say that $(C^*, d, \{F^pC^*\}_{p\geq 0})$ is a filtered differential graded module in $\mathcal{A}$.

For a filtered differential graded module $(C^*, d, \{F^pC^*\}_{p\geq 0})$, we can construct an associated spectral sequence

$$E_1^{p,q}(C^*) \Longrightarrow H^{p+q}(C^*)$$

with

$$d_r^{p,q} : E_r^{p,q}(C^*) \longrightarrow E_r^{p+r,q-r+1}(C^*)$$

(see, for example, [8, Theorem 2.6]). Note that we have an isomorphism

$$E_1^{p,q}(C^*) \cong H^{p+q}(F^pC^*/F^{p+1}C^*).$$

The differential $d_1 : E_1^{p,q}(C^*) \to E_1^{p+1,q}(C^*)$ is identified with the connecting homomorphism

$$H^{p+q}(F^pC^*/F^{p+1}C^*) \longrightarrow H^{p+q+1}(F^{p+1}C^*/F^{p+2}C^*)$$

associated to the short exact sequence

$$0 \longrightarrow F^{p+1}C^*/F^{p+2}C^* \longrightarrow F^pC^*/F^{p+2}C^* \longrightarrow F^pC^*/F^{p+1}C^* \longrightarrow 0$$

of cochain complexes. If $F^tC^* = 0$, then the spectral sequence collapses from the E_t-page.

Now, we suppose that $\mathcal{A}$ is an abelian monoidal category, in which the tensor product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is right exact separately in each variable.

Definition 3.1. Let (A^*, d) be a differential graded algebra in $\mathcal{A}$. Suppose that we have a filtration

$$A^* = F^0A^* \supset F^1A^* \supset \cdots \supset F^nA^* \supset \cdots \supset F^tA^* = 0.$$

A triple $(A^*, d, \{F^pA^*\}_{p\geq 0})$ is said to be a filtered differential graded algebra if it satisfies the following two conditions:

(1) For any $p \geq 0$, $d(F^pA^*) \subset F^pA^*$.
(2) For any $r, s \geq 0$, $F^rA^* \cdot F^sA^* \subset F^{r+s}A^*$.

To a filtered differential graded algebra $(A^*, d, \{F^pA^*\}_{p\geq 0})$, there is a spectral sequence

$$E_1^{p,q} = H^{p+q}(F^pA/F^{p+1}A) \Longrightarrow H^{p+q}(A)$$

of algebras in $\mathcal{A}$, which converges to $H^{p+q}(A)$ as an algebra (see, for example, [8, Theorem 2.14]).

3.2. **Filtrations and spectral sequences on Hochschild cochain complexes.** In this subsection we consider filtrations on Hochschild complexes and associated spectral sequences.

Let R be a commutative ring and let A be an associative algebra over R. We assume that A is a projective module over R. For an A-bimodule M over R, we denote by $C^*(A, M)$ the Hochschild cochain complex.

First, we suppose that there exists a filtration of A-bimodules over R:

$$M = F^0M \supset F^1M \supset \cdots \supset F^pM \supset \cdots \supset F^tM = 0.$$

We denote by $\mathrm{Gr}^p(M)$ the p-th associated graded module $F^pM/F^{p+1}M$. Using the filtration $\{F^pM\}_{p\geq 0}$ on M, we can introduce a filtration $\{F^pC^*(A, M)\}_{p\geq 0}$ on $C^*(A, M)$ by

$$F^pC^*(A, M) = C^*(A, F^pM).$$

Hence we obtain the following proposition.

Proposition 3.2. *For an A-bimodule M over R equipped with a filtration $M = F^0M \supset F^1M \supset \cdots \supset F^pM \supset \cdots \supset F^tM = 0$, there exists a spectral sequence*

$$E_1^{p,q}(A, M) \cong \mathrm{HH}^{p+q}(A, \mathrm{Gr}^p(M)) \Longrightarrow \mathrm{HH}^{p+q}(A, M)$$

of R-modules with

$$d_r : E_r^{p,q}(A, M) \longrightarrow E_r^{p+r,q-r+1}(A, M)$$

for $r \geq 1$, where $\mathrm{Gr}^p(M) = F^pM/F^{p+1}M$.

In particular, we consider a filtration on the A-bimodule A by powers of a two-sided ideal. Let J be a two-sided ideal of A. We assume that $J^t = 0$ for some $t > 0$. By setting

$$F^p A = J^p$$

for $p \geq 0$, there is a filtration

$$A = F^0 A \supset F^1 A \supset \cdots \supset F^p A \supset \cdots \supset F^t A = 0$$

of A by A-bimodules over R. From this filtration, we obtain a filtration $\{C^*(A, J^p)\}_{p\geq 0}$ on $C^*(A, A)$. Recall that $C^*(A, A)$ is a differential graded algebra over R. We can easily verify that

$$C^*(A, J^r) \cdot C^*(A, J^s) \subset C^*(A, J^{r+s}).$$

Thus, the triple $(C^*(A, A), d, \{C^*(A, J^r)\}_{r\geq 0})$ is a filtered differential graded algebra over R, and we obtain the following proposition.

Proposition 3.3. *There is a spectral sequence of R-algebras*

$${}^J E_1^{p,q}(A, A) \Longrightarrow \mathrm{HH}^{p+q}(A, A),$$

where

$${}^J E_1^{p,q}(A, A) \cong \mathrm{HH}^{p+q}(A, J^p/J^{p+1}).$$

Now, we consider the induced filtration on A-bimodules over R. Let M be an A-bimodule over R. By setting

$$\overline{J}^p M = \sum_{a+b=p} J^a M J^b \tag{3.1}$$

for $p \geq 0$, we obtain a filtration

$$M = \overline{J}^0 M \supset \overline{J}^1 M \supset \cdots \supset \overline{J}^p M \supset \cdots \supset \overline{J}^{2t-1} M = 0$$

of M by A-bimodules over R. From this filtration, we obtain a filtration $\{C^*(A, \overline{J}^p M)\}_{p\geq 0}$ on $C^*(A, M)$ and the following proposition.

Proposition 3.4. *There is a spectral sequence of R-modules*

$${}^J E_1^{p,q}(A, M) \Longrightarrow \mathrm{HH}^{p+q}(A, M).$$

We have an isomorphism

$${}^J E_1^{p,q}(A, M) \cong \mathrm{HH}^{p+q}(A, \mathrm{Gr}_J^p(M)),$$

where

$$\mathrm{Gr}_J^p(M) = \overline{J}^p M / \overline{J}^{p+1} M.$$

Remark 3.5. Since

$$C^*(A, J^a) \cdot C^*(A, \overline{J}^b M) \cdot C^*(A, J^c) \subset C^*(A, \overline{J}^{a+b+c} M),$$

the triple $(C^*(A, M), d, \{C^*(A, \overline{J}^p M)\}_{p\geq 0})$ is a differential graded bimodule over the differential graded algebra $(C^*(A, A), d, \{C^*(A, J^p)\}_{p\geq 0})$. Thus, the spectral sequence $\{{}^J E_r^{*,*}(A, M), d_r\}_{r\geq 1}$ is a bimodule over the spectral sequence $\{{}^J E_r^{*,*}(A, A), d_r\}_{r\geq 1}$.

3.3. **Gradings on spectral sequences.** Let $\mathcal{A}$ be an abelian category with countable coproducts. We let $\mathcal{A}^{\mathbb{Z}}$ be the abelian category of $\mathbb{Z}$-graded objects of $\mathcal{A}$ and grading-preserving morphisms. In this subsection we consider spectral sequences in $\mathcal{A}^{\mathbb{Z}}$. In this case we have trigradings on spectral sequences.

Let

$$C^* = \bigoplus_{s\in\mathbb{Z}} C^{*,s}$$

be a cochain complex of $\mathcal{A}^{\mathbb{Z}}$ equipped with a filtration

$$C^* = F^0C^* \supset F^1C^* \supset \cdots \supset F^pC^* \supset \cdots \supset F^tC^* = 0$$

by subcomplexes, where $C^{*,s}$ is the component of (cohomological) degree $s \in \mathbb{Z}$. Set $F^pC^{n,s} = F^pC^n \cap C^{n,s}$. Note that any filtration by subcomplexes in $\mathcal{A}^{\mathbb{Z}}$ is assumed to satisfy $F^pC^* = \bigoplus_{s\in\mathbb{Z}} F^pC^{*,s}$ (in other words, the filtration is compatible with the grading). Then we have the following associated spectral sequence as in §3.1.

Proposition 3.6. *There is a spectral sequence*

$$E_1^{p,q}(C^*) \Longrightarrow H^{p+q}(C^*)$$

in $\mathcal{A}^{\mathbb{Z}}$*, with*

$$E_1^{p,q}(C^*) \cong H^{p+q}(F^pC^*/F^{p+1}C^*).$$

More precisely, let $E_r^{p,q,s}(C^*)$ *be the degree* s *component of the spectral sequence* $\{E_r^{p,q}(C^*), d_r^{p,q}\}_{r\geq 1}$*. Then* $\{E_r^{p,q,s}(C^*), d_r^{p,q,s}\}_{r\geq 1}$ *is a spectral sequence of* $\mathcal{A}$ *with*

$$d_r^{p,q,s} : E_r^{p,q,s}(C^*) \longrightarrow E_r^{p+r,q-r+1,s}(C^*),$$

where

$$E_r^{p,q}(C^*) = \bigoplus_{s\in\mathbb{Z}} E_r^{p,q,s}(C^*),\ d_r^{p,q} = \bigoplus_{s\in\mathbb{Z}} d_r^{p,q,s},$$

and

$$E_1^{p,q,s}(C^*) \cong H^{p+q}(F^pC^{*,s}/F^{p+1}C^{*,s}) \Longrightarrow H^{p+q}(C^{*,s}).$$

Notation 3.7. Under the situation above, we set

$$H^{n,s}(C^*) = H^n(C^{*,s}).$$

Note that

$$H^n(C^*) = \bigoplus_{s\in\mathbb{Z}} H^{n,s}(C^*).$$

We need the following lemma in §5 below.

Lemma 3.8. *If there exists an integer* i *such that*

$$H^{p+q,s}(F^pC^*/F^{p+1}C^*) = H^{p+q}(F^pC^{*,s}/F^{p+1}C^{*,s}) = 0$$

for $s \neq q+i$*, then the spectral sequence*

$$E_1^{p,q}(C^*) \Longrightarrow H^{p+q}(C^*)$$

collapses from the E_2*-page.*

Proof. By the assumption, $E_1^{p,q,s}(C^*) = 0$ for $s \neq q+i$. Since the differential of the spectral sequence $\{E_r^{p,q,s}(C^*), d_r\}_{r\geq 1}$ has the form $d_r^{p,q,s} : E_r^{p,q,s}(C^*) \to E_r^{p+r,q-r+1,s}(C^*)$, we see that d_r is trivial unless $r = 1$. Hence the spectral sequence collapses from the E_2-page. □

4. $\mathrm{HH}^*(\mathrm{N}_m(R), R)$

In this section, we show that $\mathrm{HH}^*(\mathrm{N}_m(R), R) \cong \mathrm{N}_m(R)^!$ as graded R-algebras. We also obtain several results on $\varphi(n) = \mathrm{rank}_R \mathrm{N}_m(R)^!_n$. By using these results, it is possible to calculate $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ and $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ as R-modules. Indeed, $\mathrm{M}_m(R)/\mathrm{N}_m(R)$ and $\mathrm{N}_m(R)$ have filtrations of $\mathrm{N}_m(R)$-bimodules over R whose associated graded modules are isomorphic to direct sums of copies of $\mathrm{N}_m(R)$-bimodules R over R. By calculating spectral sequences, we will determine the R-module structure of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ in §5 and that of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ in §6, respectively.

In §4.1, we deal with quadratic monomial algebras A over a commutative ring R. We show that $\mathrm{HH}^*(A, R) \cong A^!$ as graded R-algebras. In §4.2, we apply the results in §4.1 to the case $A = \mathrm{N}_m(R)$ and determine the R-algebra structure of $\mathrm{HH}^*(\mathrm{N}_m(R), R)$ for $m \geq 2$. In §4.3, we obtain several results on $\varphi(n) = \mathrm{rank}_R \mathrm{N}_m(R)^!_n$.

4.1. **Quadratic monomial algebras.** In this subsection, we deal with quadratic monomial algebras over a commutative ring R (cf. [11, Chapter 1 §2]). Let $\{e_1, \dots, e_n\}$ be an R-basis of a free R-module V of rank n. Let $T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ be the tensor algebra of V over R. For a subset S of $\{e_i \otimes e_j \in V \otimes_R V \mid 1 \leq i, j \leq n\}$, set $I_S = R\{S\} \subseteq V \otimes_R V$. Then we say that $A_S = T(V)/\langle I_S \rangle$ is a *quadratic monomial algebra* over R, where $\langle I_S \rangle$ is the two-sided ideal of $T(V)$ generated by I_S. We also write $A_S = \{V, I_S\}$ according to [11, Chapter 1 §2]. Note that A_S is a graded R-algebra with $|e_i| = 1$ $(1 \leq i \leq n)$. Denote by $A_{S,i}$ the homogenous part of A_S of degree i. Then $A_S = R \oplus A_{S,+}$ is an augmented algebra over R with augmentation map $\epsilon : A_S \to R$, where $A_{S,+} = \oplus_{i>0} A_{S,i}$ and $\epsilon(A_{S,+}) = 0$.

Let $V^* = \mathrm{Hom}_R(V, R)$. Let $\{e_1^*, \dots, e_n^*\} \subset V^*$ be the dual basis of $\{e_1, \dots, e_n\}$. Set

$$I_S^{\perp} = \{f \in V^* \otimes_R V^* \mid f(v) = 0 \text{ for any } v \in I_S\}$$

and

$$S^{\perp} = I_S^{\perp} \cap \{e_i^* \otimes e_j^* \in V^* \otimes_R V^* \mid 1 \leq i, j \leq n\}.$$

Then $I_S^{\perp} = R\{S^{\perp}\} \subseteq V^* \otimes_R V^*$. We define the *quadratic dual algebra* $A_S^!$ of A_S by

$$A_S^! = T(V^*)/\langle I_S^{\perp} \rangle,$$

where $\langle I_S^{\perp} \rangle$ is the two-sided ideal of $T(V^*)$ generated by $I_S^{\perp}$. The quadratic monomial algebra $A_S^! = \{V^*, I_S^{\perp}\}$ over R is a graded R-algebra with $|e_i^*| = 1$ $(1 \leq i \leq n)$.

Put $A = A_S$, $A^! = A_S^!$, $I = I_S$, and $I^{\perp} = I_S^{\perp}$. The degree d part $A_d^!$ of $A^!$ can be described by

$$A_d^! = V^{*\otimes d} / \Big(\sum_{i+j=d-2} V^{*\otimes i} \otimes_R I^{\perp} \otimes_R V^{*\otimes j} \Big).$$

Thus, we can write the dual module $(A_d^!)^* = \mathrm{Hom}_R(A_d^!, R)$ of $A_d^!$ by

$$(A_d^!)^* = \bigcap_{i+j=d-2} V^{\otimes i} \otimes_R I \otimes_R V^{\otimes j} \subseteq V^{\otimes d}.$$

We define the complex $\widehat{K}_*(A)$ of graded free A-bimodules by

$$\cdots \longrightarrow A \otimes_R (A_3^!)^* \otimes_R A \xrightarrow{\widehat{d}_3} A \otimes_R (A_2^!)^* \otimes_R A \xrightarrow{\widehat{d}_2} A \otimes_R (A_1^!)^* \otimes_R A \xrightarrow{\widehat{d}_1} A \otimes_R A \longrightarrow 0,$$

where $\widehat{d}_i : A \otimes_R (A^!_i)^* \otimes_R A \to A \otimes_R (A^!_{i-1})^* \otimes_R A$ is defined by

$$\sum_\lambda a \otimes a_1^\lambda \otimes a_2^\lambda \otimes \cdots \otimes a_i^\lambda \otimes b \longmapsto \sum_\lambda aa_1^\lambda \otimes a_2^\lambda \otimes \cdots \otimes a_i^\lambda \otimes b + (-1)^i \sum_\lambda a \otimes a_1^\lambda \otimes a_2^\lambda \otimes \cdots \otimes a_i^\lambda b. \tag{4.1}$$

Note that $\widehat{K}_*(A)$ is a subcomplex of the reduced bar complex $\overline{B}_*(A, A, A)$. We also define the Koszul complex $K_*(A)$ of graded free A-modules by

$$\cdots \longrightarrow A \otimes_R (A^!_3)^* \xrightarrow{d_3} A \otimes_R (A^!_2)^* \xrightarrow{d_2} A \otimes_R (A^!_1)^* \xrightarrow{d_1} A \longrightarrow 0,$$

where $d_i : A \otimes_R (A^!_i)^* \to A \otimes_R (A^!_{i-1})^*$ is defined by

$$\sum_\lambda a \otimes a_1^\lambda \otimes a_2^\lambda \otimes \cdots \otimes a_n^\lambda \longmapsto \sum_\lambda aa_1^\lambda \otimes a_2^\lambda \otimes \cdots \otimes a_n^\lambda.$$

Then $K_*(A) \cong \widehat{K}_*(A) \otimes_A R$.

Using $A = \oplus_{i\geq 0} A_i$, we obtain the following decompositions:

$$\widehat{K}_d(A) = A \otimes_R (A^!_d)^* \otimes_R A = \bigoplus_{l\geq 0} \widehat{K}_d(A)_l \quad \text{and} \quad K_d(A) = A \otimes_R (A^!_d)^* = \bigoplus_{l\geq 0} K_d(A)_l$$

for $d \geq 0$, where

$$\widehat{K}_d(A)_l = \bigoplus_{i+d+j=l} A_i \otimes_R (A^!_d)^* \otimes_R A_j \quad \text{and} \quad K_d(A)_l = \bigoplus_{i+d=l} A_i \otimes_R (A^!_d)^*.$$

Here $(A^!_d)^* (\subseteq V^{\otimes d})$, $\widehat{K}_d(A)_l$, and $K_d(A)_l$ are equipped with internal degree d, l, l, respectively. Note that $\widehat{K}_*(A) = \oplus_{l\geq 0}\widehat{K}_*(A)_l$ and $K_*(A) = \oplus_{l\geq 0} K_*(A)_l$ are the direct sums of subcomplexes.

By the augmentation map $\epsilon : A \to R$, R can be considered as a left A-module (or an A-bimodule). The bar-resolution $\widetilde{Bar}_*(A, R)$ of the left A-module R is

$$\cdots \xrightarrow{\partial_3} A \otimes_R A_+^{\otimes 2} \xrightarrow{\partial_2} A \otimes_R A_+ \xrightarrow{\partial_1} A \xrightarrow{\epsilon} R \longrightarrow 0,$$

where $\widetilde{Bar}_i(A, R) = A \otimes_R A_+^{\otimes i}$ and the differential $\partial_i : \widetilde{Bar}_i(A, R) \to \widetilde{Bar}_{i-1}(A, R)$ is given by

$$\partial_i(a_0 \otimes a_1 \otimes \cdots \otimes a_i) = \sum_{j=0}^{i-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i.$$

Let us consider the cochain complex $Cob^*(A) = \mathrm{Hom}_A(\widetilde{Bar}_*(A, R), R)$: for $i \geq 0$, we have

$$Cob^i(A) = \mathrm{Hom}_A(\widetilde{Bar}_i(A, R), R) \cong A_+^{\otimes i} \quad (i > 0), \quad Cob^0(A) = R,$$

$$Cob^i(A) = \bigoplus_{j\geq i} Cob^{ij}(A), \quad Cob^{ij}(A) = \bigoplus_{k_1+\cdots+k_i=j, k_s\geq 1} A_{k_1} \otimes_R \cdots \otimes_R A_{k_i} \quad (i > 0),$$

$$Cob^0(A) = Cob^{00}(A) = R, \quad Cob^{0j}(A) = 0 \quad (j > 0).$$

Then $\mathrm{Ext}^i_A(R, R) = H^i(Cob^*(A)) = \oplus_j \mathrm{Ext}^{ij}_A(R, R)$ and $\mathrm{Ext}^{ij}_A(R, R) = H^i(Cob^{*,j}(A))$.

Proposition 4.1. *Let $A = \{V, I_S\}$ be the quadratic monomial algebra over a commutative ring R associated to a subset S of $\{e_i \otimes e_j \in V \otimes_R V \mid 1 \leq i, j \leq n\}$. Then*

1. $\mathrm{Ext}^{ij}_A(R, R) = 0$ *for $i \neq j$.*
2. $\mathrm{Ext}^*_A(R, R) \cong A^!$ *as graded R-algebras.*
3. $H_i(\widehat{K}_*(A)) = 0$ $(i > 0)$ *and* $H_0(\widehat{K}_*(A)) = A$.
4. $H_i(K_*(A)) = 0$ $(i > 0)$ *and* $H_0(K_*(A)) = R$.

(5) *R admits a linear minimal graded free resolution as an A-module over R, in other words, there exists a graded free resolution over R*

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} R \longrightarrow 0$$

such that $d_i : P_i = A \otimes_R X_i \to P_{i-1} = A \otimes_R X_{i-1}$ can be described by a matrix whose entries are in A_1 with respect to R-bases of X_i and X_{i-1}, where X_i is a graded free R-module for each $i \geq 0$.

Proof. Let $X_i = V^{\otimes i-1} \otimes_R I_S \otimes_R V^{\otimes d-i-1} \subset V^{\otimes d}$ for $1 \leq i \leq d-1$. Since $I_S = R\{S\}$, the collection $(X_1, \ldots, X_{d-1})$ is distributive in the lattice $(V^{\otimes d}, \cap, \cup)$ consisting of free R-submodules by the same discussion in the proof of [11, Chapter 1 Proposition 7.1] $(c) \Longrightarrow (a)$. As in the proof of [11, Chapter 2 Theorem 4.1], we see that (1) $\mathrm{Ext}_A^{ij}(R, R) = 0$ for $i \neq j$ and (4) $H_i(K_*(A)) = 0$ $(i > 0)$ and $H_0(K_*(A)) = R$ hold. We can also prove that (2) $\mathrm{Ext}_A^*(R, R) \cong A^!$ as graded R-algebras in the same way as [11, Chapter 1 Proposition 3.1]. Since $K_*(A)$ gives a linear minimal graded free resolution of R, (5) holds.

Set $\mathcal{I} = \{(i,j) \in [1, \ldots, n] \times [1, \ldots, n] \mid e_i \otimes e_j \in S\}$. Let $\{e'_1, \ldots, e'_n\}$ be a $\mathbb{Z}$-basis of a free $\mathbb{Z}$-module $V_\mathbb{Z}$. Put $S_\mathbb{Z} = \{e'_i \otimes e'_j \in V_\mathbb{Z} \otimes_\mathbb{Z} V_\mathbb{Z} \mid (i,j) \in \mathcal{I}\}$, $I_{S_\mathbb{Z}} = \mathbb{Z}\{S_\mathbb{Z}\} \subseteq V_\mathbb{Z} \otimes_\mathbb{Z} V_\mathbb{Z}$, and $A_\mathbb{Z} = T(V_\mathbb{Z})/\langle I_{S_\mathbb{Z}} \rangle$. Then $A_\mathbb{Z}$ is a quadratic monomial algebra over $\mathbb{Z}$ and $A = A_\mathbb{Z} \otimes_\mathbb{Z} R$. Note that $\widehat{K}_*(A) \cong \widehat{K}_*(A_\mathbb{Z}) \otimes_\mathbb{Z} R$ and $K_*(A) \cong K_*(A_\mathbb{Z}) \otimes_\mathbb{Z} R$. To show that (3) holds for $\widehat{K}_*(A)$, it suffices to prove that (3) holds for $\widehat{K}_*(A_\mathbb{Z})$ by Künneth theorem. For any field k, $A_k = A_\mathbb{Z} \otimes_\mathbb{Z} k$ is a quadratic monomial algebra over k. By [14, Theorem 3.4.6], (3) holds for the complex $\widehat{K}_*(A_k) = \widehat{K}_*(A_\mathbb{Z}) \otimes_\mathbb{Z} k$ of A_k-bimodules over k. Each internal degree component of $\widehat{K}_*(A_\mathbb{Z})$ is a complex of finitely generated free modules over $\mathbb{Z}$. Using Lemma 4.2, (3) holds for $\widehat{K}_*(A_\mathbb{Z})$. (Note that we can prove (4) holds for $K_*(A)$ in the same way.) □

The following lemma has been used in the proof of Proposition 4.1.

Lemma 4.2. *Let C_* be a complex of finitely generated free modules over $\mathbb{Z}$. If $C_* \otimes_\mathbb{Z} k$ is acyclic for any field k, then C_* is acyclic.*

Proof. By the assumption, $H_i(C_* \otimes_\mathbb{Z} k) = 0$ for any $i \in \mathbb{Z}$. Using Künneth theorem, we have an exact sequence

$$0 \longrightarrow H_i(C_*) \otimes_\mathbb{Z} k \longrightarrow H_i(C_* \otimes_\mathbb{Z} k) \longrightarrow \mathrm{Tor}_1(H_{i-1}(C_*), k) \longrightarrow 0,$$

which implies that $H_i(C_*) \otimes_\mathbb{Z} k = 0$ for any field k. Since $H_i(C_*)$ is a finitely generated module over $\mathbb{Z}$, $H_i(C_*)$ must be 0. Hence C_* is acyclic. □

Remark 4.3. In Proposition 4.1, we can prove that (4) implies (3) for $\widehat{K}_*(A_k)$ with a field k in the following way. By [11, Chapter 1 Proposition 4.2], there exists a decomposition $\widehat{K}_*(A_k) = P_* \oplus T_*$ into the direct sum of two subcomplexes of free graded A_k-bimodules, where P_* is minimal and T_* is acyclic. Here we say that P_* is *minimal* if the induced map $P_{i+1} \otimes_{A_k} k \to P_i \otimes_{A_k} k$ vanish for any $i \in \mathbb{Z}$. Using $\widehat{K}_*(A_k) \otimes_{A_k} k = K_*(A_k)$, we have

$$H_i(K_*(A_k)) = H_i(P_* \otimes_{A_k} k) \oplus H_i(T_* \otimes_{A_k} k) = \begin{cases} 0 & (i \neq 0), \\ k & (i = 0). \end{cases}$$

Thus, we obtain

$$P_i \otimes_k k = \begin{cases} 0 & (i \neq 0), \\ k & (i = 0). \end{cases}$$

Since P_* has finite-dimensional grading components, $P_i = 0$ for $i \neq 0$ by Nakayama's lemma for noncommutative graded algebras ([11, Chapter 1 Lemma 4.1]). Hence, $H_i(\widehat{K}_*(A_k)) = H_i(T_*) = 0$ for $i > 0$. We also see that $H_0(\widehat{K}_*(A_k)) = k$ directly.

Remark 4.4. Let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis of V^* of an R-basis $\{e_1, \ldots, e_n\}$ of a free R-module V. Let $\phi : V \to V^*$ be the R-isomorphism defined by $e_i \mapsto e_i^*$ $(1 \le i \le n)$. Set $A = \{V, I_S\}$ and $A^! = \{V^*, I_S^{\perp}\}$. Then ϕ induces an R-isomorphism $(A_d^!)^* \to A_d^!$ for $d \ge 0$ and an isomorphism of chain complexes of A-bimodules over R:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & A \otimes_R (A_2^!)^* \otimes_R A & \xrightarrow{\widehat{d_2}} & A \otimes_R (A_1^!)^* \otimes_R A & \xrightarrow{\widehat{d_1}} & A \otimes_R A & \xrightarrow{\mu} & A & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & A \otimes_R A_2^! \otimes_R A & \longrightarrow & A \otimes_R A_1^! \otimes_R A & \longrightarrow & A \otimes_R A & \xrightarrow{\mu} & A & \longrightarrow & 0,
\end{array}$$

where $\mu : A \otimes_R A \to A$ is defined by $\mu(a \otimes b) = ab$ and the second exact row is given in [12, Theorem 3]. In other words, $K_*(A)$ gives us the free resolution of A-bimodules of A over R which is isomorphic to the one in [12, Theorem 3].

Theorem 4.5. *Let $A = \{V, I_S\}$ be the monomial quadratic algebra over a commutative ring R associated to a subset S of $\{e_i \otimes e_j \in V \otimes_R V \mid 1 \le i, j \le n\}$. Then $\mathrm{HH}^*(A, R) \cong A^!$ as graded R-algebras.*

Proof. By Proposition 4.1 (3), $\widehat{K}_*(A)$ gives us a graded free resolution of A as A-bimodules over R. By taking $\mathrm{Hom}_{A^e}(-, R)$ of $\widehat{K}_*(A)$, we obtain a cochain complex whose differentials are all 0. Hence, $\mathrm{HH}^i(A, R) = H^i(\mathrm{Hom}_{A^e}(\widehat{K}_*(A), R)) = \mathrm{Hom}_{A^e}(\widehat{K}_i(A), R) \cong \mathrm{Hom}_R((A_i^!)^*, R) \cong A_i^!$.

Let us show that $\mathrm{HH}^*(A, R) \cong A^!$ as graded R-algebras. Since $\widehat{K}_*(A)$ is a subcomplex of the reduced bar complex $\overline{B}_*(A, A, A)$, we have a morphism of chain complexes $f = (f_i)$:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & A \otimes_R (A_2^!)^* \otimes_R A & \xrightarrow{\widehat{d_2}} & A \otimes_R (A_1^!)^* \otimes_R A & \xrightarrow{\widehat{d_1}} & A \otimes_R A & \xrightarrow{\mu} & A & \longrightarrow & 0 \\
& \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \| & & \\
\cdots \longrightarrow & A \otimes_R \overline{A}^{\otimes 2} \otimes_R A & \longrightarrow & A \otimes_R \overline{A} \otimes_R A & \longrightarrow & A \otimes_R A & \xrightarrow{\mu} & A & \longrightarrow & 0,
\end{array}$$

where $\overline{A} = A/RI$ and I is the image of $1 \in R$ under the unit map $R \to A$. By taking $\mathrm{Hom}_{A^e}(-, R)$ of chain complexes, we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{A^e}(A \otimes_R A, R) & \longrightarrow & \mathrm{Hom}_{A^e}(A \otimes_R \overline{A} \otimes_R A, R) & \longrightarrow & \cdots \\
& & \downarrow f_0^* & & \downarrow f_1^* & & \\
0 & \longrightarrow & \mathrm{Hom}_{A^e}(A \otimes_R A, R) & \xrightarrow{\widehat{d_1}^*} & \mathrm{Hom}_{A^e}(A \otimes_R (A_1^!)^* \otimes_R A, R) & \xrightarrow{\widehat{d_2}^*} & \cdots,
\end{array}$$

which are isomorphic to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{d^0} & \mathrm{Hom}_R(\overline{A}, R) & \xrightarrow{d^1} & \mathrm{Hom}_R(\overline{A} \otimes_R \overline{A}, R) & \xrightarrow{d^2} & \cdots \\
& & \downarrow id & & \downarrow f_1^* & & \downarrow f_2^* & & \\
0 & \longrightarrow & R & \xrightarrow{0} & A_1^! & \xrightarrow{0} & A_2^! & \xrightarrow{0} & \cdots.
\end{array} \tag{4.2}$$

This is a quasi-isomorphism of cochain complexes. For $g_m = e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* \in \mathrm{Hom}_R(\overline{A}^{\otimes m}, R)$ and $g_d = e_{j_1}^* \otimes \cdots \otimes e_{j_d}^* \in \mathrm{Hom}_R(\overline{A}^{\otimes d}, R)$, we have $f_m^*(g_m) = e_{i_1}^* \cdots e_{i_m}^* \in A_m^!$ and $f_d^*(g_d) = e_{j_1}^* \cdots e_{j_d}^* \in A_d^!$. The restriction of the product $g_m \cdot g_d = e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* \otimes e_{j_1}^* \otimes \cdots \otimes e_{j_d}^* \in \mathrm{Hom}_R(\overline{A}^{\otimes(m+d)}, R)$ to $(A_{m+d}^!)^*$ is equal to $e_{i_1}^* \cdots e_{i_m}^* e_{j_1}^* \cdots e_{j_d}^* \in A_{m+d}^!$. This implies that $\mathrm{HH}^*(A, R) \cong A^!$ as graded R-algebras. □

Proposition 4.6. *Let $A = \{V, I_S\}$ be the monomial quadratic algebra over a commutative ring R. Let L be an A-bimodule over R. Assume that $A_+L = LA_+ = 0$, where $A_+ = \oplus_{d>0} A_d$. Then $\mathrm{HH}^i(A, L) \cong \mathrm{HH}^i(A, R) \otimes_R L \cong A_i^! \otimes_R L$ for $i \ge 0$.*

Proof. By Proposition 4.1 (3), $\widehat{K}_*(A)$ gives us a graded free resolution of A as A-bimodules over R. Since $A_+L = LA_+ = 0$, we obtain a cochain complex with zero differential

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A\otimes_R A, L) \xrightarrow{0} \mathrm{Hom}_{A^e}(A\otimes_R (A^!_1)^*\otimes_R A, L) \xrightarrow{0} \mathrm{Hom}_{A^e}(A\otimes_R (A^!_2)^*\otimes_R A, L) \xrightarrow{0} \cdots$$

by taking $\mathrm{Hom}_{A^e}(-, L)$ of $\widehat{K}_*(A)$. This is isomorphic to

$$0 \longrightarrow L \xrightarrow{0} A^!_1 \otimes_R L \xrightarrow{0} A^!_2 \otimes_R L \xrightarrow{0} \cdots .$$

Hence

$$\mathrm{HH}^i(A, L) \cong A^!_i \otimes_R L \cong \mathrm{HH}^i(A, R) \otimes_R L$$

for each $i \geq 0$. □

Suppose that $A = \{V, I_S\}$ has finite rank over R. Set $\overline{A} = A/RI$, where $I \in A$ is the image of $1 \in R$ under the unit map $R \to A$. Denote by $\overline{A}_d$ the degree d component of $\overline{A}$. Note that $\overline{A} = \bigoplus_{d\geq 1} \overline{A}_d$. Let $\overline{B}_p(R, A, R) = \overbrace{\overline{A} \otimes_R \cdots \otimes_R \overline{A}}^{p}$ $(p > 0)$ and $\overline{B}_0(R, A, R) = R$. For $p > 0$, set

$$\overline{B}_p(R, A, R)_d = \bigoplus_{a_1+\cdots+a_p=d} \overline{A}_{a_1} \otimes_R \cdots \otimes_R \overline{A}_{a_p}.$$

For $p = 0$, set

$$\overline{B}_0(R, A, R)_d = \begin{cases} R & (d = 0), \\ 0 & (d \neq 0). \end{cases}$$

Let $\overline{C}^*(A, R)$ and $\overline{C}^{*,d}(A, R)$ be the cochain complexes defined by $\overline{C}^p(A, R) = \mathrm{Hom}_R(\overline{B}_p(R, A, R), R)$ and $\overline{C}^{p,d}(A, R) = \mathrm{Hom}_R(\overline{B}_p(R, A, R)_d, R)$, respectively. The differential $d^p : \overline{C}^p(A, R) \to \overline{C}^{p+1}(A, R)$ (resp. $d^p : \overline{C}^{p,d}(A, R) \to \overline{C}^{p+1,d}(A, R)$) is defined by

$$d^p(f)(a_1 \otimes \cdots \otimes a_{p+1}) = \sum_{j=1}^{p} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{p+1})$$

for $f \in \overline{C}^p(A, R)$ (resp. $f \in \overline{C}^{p,d}(A, R)$). Since $\mathrm{rank}_R A < \infty$, we have

$$\overline{C}^p(A, R) = \bigoplus_{d\in\mathbb{Z}} \overline{C}^{p,d}(A, R).$$

Denoting by $\mathrm{HH}^{p,d}(A, R)$ the p-th cohomology of $\overline{C}^{*,d}(A, R)$ as in Notation 3.7, we have

$$\mathrm{HH}^p(A, R) = \bigoplus_{d\in\mathbb{Z}} \mathrm{HH}^{p,d}(A, R).$$

Theorem 4.7. *Suppose that $A = \{V, I_S\}$ has finite rank over R. For $p \geq 0$, we have*

$$\mathrm{HH}^{p,d}(A, R) \cong \begin{cases} A^!_p & (d = p), \\ 0 & (d \neq p). \end{cases}$$

Proof. By Theorem 4.5, $\mathrm{HH}^p(A, R) \cong A^!_p$ for $p \geq 0$. By (4.2), it can be verified that $A^!_p \subseteq \mathrm{HH}^{p,p}(A, R)$. Hence, $\mathrm{HH}^{p,d}(A, R) = 0$ for $d \neq p$ and $\mathrm{HH}^{p,p}(A, R) \cong A^!_p$. □

4.2. **The R-algebra structure of** $\mathrm{HH}^*(\mathrm{N}_m(R), R)$**.** In this subsection, we apply the results in §4.1 to the case $A = \mathrm{N}_m(R)$ and determine the R-algebra structure of $\mathrm{HH}^*(\mathrm{N}_m(R), R)$ for $m \geq 2$. Let R be a commutative ring. For $m \geq 2$, set

$$\mathrm{N}_m(R) = \left\{ \begin{pmatrix} a & * & * & \cdots & * \\ 0 & a & * & \cdots & * \\ 0 & 0 & a & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in \mathrm{M}_m(R) \right\}.$$

Putting

$$x_1 = E_{1,2},\ x_2 = E_{2,3},\ \ldots,\ x_{m-1} = E_{m-1,m} \in \mathrm{N}_m(R),$$

we have an isomorphism $\mathrm{N}_m(R) \cong R\langle x_1, x_2, \ldots, x_{m-1}\rangle / \langle x_i x_j \mid j \neq i+1\rangle$. Let $V = Rx_1 \oplus \cdots \oplus Rx_{m-1}$ be a free R-module of rank $m-1$. Set $S = \{x_i \otimes x_j \mid j \neq i+1\} \subset V \otimes V$. Then $\mathrm{N}_m(R) = \{V, I_S\}$ is a quadratic monomial algebra over R with $|x_i| = 1$.

We define the two-sided ideal $J(\mathrm{N}_m(R))$ of $\mathrm{N}_m(R)$ over R by

$$J(\mathrm{N}_m(R)) = \{(a_{ij}) \in \mathrm{N}_m(R) \mid a_{11} = a_{22} = \cdots = a_{nn} = 0\}.$$

Note that $J(\mathrm{N}_m(R)) = \mathrm{N}_m(R)_+ = \oplus_{i>0}\mathrm{N}_m(R)_i$. When R is a field, $J(\mathrm{N}_m(R))$ is the Jacobson radical of $\mathrm{N}_m(R)$. We denote $(I_m \bmod J(\mathrm{N}_m(R))) \in \mathrm{N}_m(R)/J(\mathrm{N}_m(R))$ by e. Then we see that $\mathrm{N}_m(R)/J(\mathrm{N}_m(R)) = Re \cong R$ as an $\mathrm{N}_m(R)$-bimodule over R. In the sequel, the $\mathrm{N}_m(R)$-bimodule R over R means $\mathrm{N}_m(R)/J(\mathrm{N}_m(R)) = Re$.

Put $\mathrm{N} = \mathrm{N}_m(R)$ and $\mathrm{J} = J(\mathrm{N}_m(R))$. Denote by $\mathrm{N}^! = \mathrm{N}_m(R)^!$ the quadratic dual algebra of N. Note that $S^\perp = \{x_i^* \otimes x_{i+1}^* \mid 1 \leq i \leq m-2\}$, where $\{x_1^*, \ldots, x_{m-1}^*\}$ is the dual basis of $V^* = \mathrm{Hom}_R(V, R)$ of the R-basis $\{x_1, \ldots, x_{m-1}\}$ of V. Setting $y_i = x_i^*$ $(1 \leq i \leq m-1)$, we can write $\mathrm{N}^! = R\langle y_1, y_2, \ldots, y_{m-1}\rangle / \langle y_i y_{i+1} \mid 1 \leq i < m-1\rangle$ with $|y_i| = 1$. Let us denote by $\mathrm{N}^!_n$ the homogeneous part of $\mathrm{N}^!$ of degree n. We also denote by $\mathcal{B}(\mathrm{N}^!_n)$ the R-basis of $\mathrm{N}^!_n$ consisting of monomials of degree n in $\{y_1, \ldots, y_{m-1}\}$. (Set $\mathcal{B}(\mathrm{N}^!_0) = \{1\}$.) Put $\mathcal{B}(\mathrm{N}^!) = \cup_{n=0}^\infty \mathcal{B}(\mathrm{N}^!_n)$.

Theorem 4.8. *We have an isomorphism*

$$\mathrm{HH}^*(\mathrm{N}, R) \cong \mathrm{N}^! \cong R\langle y_1, y_2, \ldots, y_{m-1}\rangle / \langle y_i y_{i+1} \mid 1 \leq i \leq m-2\rangle$$

of graded R-algebras, where $|y_i| = 1$ for $1 \leq i \leq m-1$.

Proof. The statement follows from Theorem 4.5. □

Proposition 4.9. *Let L be an* N*-bimodule over R. Assume that* $\mathrm{J}L = L\mathrm{J} = 0$*. Then* $\mathrm{HH}^n(\mathrm{N}, L) \cong \mathrm{HH}^n(\mathrm{N}, R) \otimes_R L \cong \mathrm{N}^!_n \otimes_R L$ *for $n \geq 0$.*

Proof. The statement follows from Proposition 4.6. □

Let $\overline{\mathrm{N}} = \mathrm{N}/RI_m$. Denote by $\overline{\mathrm{N}}_d$ the degree d component of $\overline{\mathrm{N}}$. Note that $\overline{\mathrm{N}} = \bigoplus_{d=1}^{m-1} \overline{\mathrm{N}}_d$. Set

$$\overline{B}_p(\mathrm{N}, \mathrm{N}, \mathrm{N}) = \mathrm{N} \otimes_R \overbrace{\overline{\mathrm{N}} \otimes_R \cdots \otimes_R \overline{\mathrm{N}}}^{p} \otimes_R \mathrm{N}$$

for $p \geq 0$. Let $\overline{B}_p(R, \mathrm{N}, R) = \overbrace{\overline{\mathrm{N}} \otimes_R \cdots \otimes_R \overline{\mathrm{N}}}^{p}$ and $\overline{B}_p(R, \mathrm{N}, R)_d = \displaystyle\bigoplus_{a_1 + \cdots + a_p = d} \overline{\mathrm{N}}_{a_1} \otimes_R \cdots \otimes_R \overline{\mathrm{N}}_{a_p}$.

Let $\overline{C}^*(\mathrm{N}, R)$ and $\overline{C}^{*,d}(\mathrm{N}, R)$ be the cochain complexes defined by $\overline{C}^p(\mathrm{N}, R) = \mathrm{Hom}_R(\overline{B}_p(R, \mathrm{N}, R), R)$

and $\overline{C}^{p,d}(\mathrm{N},R) = \mathrm{Hom}_R(\overline{B}_p(R,\mathrm{N},R)_d, R)$, respectively. Denoting by $\mathrm{HH}^{p,d}(\mathrm{N},R)$ the p-th cohomology of $\overline{C}^{*,d}(\mathrm{N},R)$ as in Notation 3.7, we have

$$\mathrm{HH}^p(\mathrm{N},R) = \bigoplus_{d=0}^{p(m-1)} \mathrm{HH}^{p,d}(\mathrm{N},R).$$

Theorem 4.10. *For $n \geq 0$, we have*

$$\mathrm{HH}^{n,d}(\mathrm{N},R) \cong \begin{cases} \mathrm{N}^!_n & (d = n), \\ 0 & (d \neq n). \end{cases}$$

Proof. The statement follows from Theorem 4.7. □

4.3. **Several results on $\varphi(n)$.** In this subsection, we prove several results on $\varphi(n) = \mathrm{rank}_R \mathrm{N}^!_n$. These results will be used in §5 and §6 for describing the ranks of Hochschild cohomology over R.

Definition 4.11. For the R-algebra $\mathrm{N}^! = \oplus_{n=0}^{\infty} \mathrm{N}^!_n$, set

$$\varphi(n) = \mathrm{rank}_R \mathrm{N}^!_n = \sharp\mathcal{B}(\mathrm{N}^!_n), \tag{4.3}$$

$$f^!(t) = \sum_{n=0}^{\infty} (\mathrm{rank}_R \mathrm{N}^!_n)\, t^n. \tag{4.4}$$

Note that $\varphi(n) = 0$ for $n < 0$.

Definition 4.12. For $1 \leq i \leq m-1$, we define

$$\begin{aligned} \mathcal{B}(\mathrm{N}^!_n)(i) &= \begin{cases} \emptyset & (n = 0), \\ \{y_i f \in \mathcal{B}(\mathrm{N}^!_n) \mid f \in \mathcal{B}(\mathrm{N}^!_{n-1})\} & (n > 0), \end{cases} \\ \psi_i(n) &= \sharp\mathcal{B}(\mathrm{N}^!_n)(i) \qquad (n \geq 0). \end{aligned}$$

Note that $\psi_i(0) = 0$ and $\psi_i(1) = 1$ for $1 \leq i \leq m-1$ and that $\varphi(n) = \sum_{i=1}^{m-1} \psi_i(n)$ for $n \geq 1$.

Proposition 4.13. *We have*

$$\begin{pmatrix} \psi_1(n+1) \\ \psi_2(n+1) \\ \vdots \\ \psi_{m-1}(n+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \psi_1(n) \\ \psi_2(n) \\ \vdots \\ \psi_{m-1}(n) \end{pmatrix} \quad (n \geq 1)$$

and

$$\begin{pmatrix} \psi_1(1) \\ \psi_2(1) \\ \vdots \\ \psi_{m-1}(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Proof. Recall that $\psi_i(1) = 1$ for $1 \leq i \leq m-1$. Note that $y_i y_{i+1} = 0$ in $\mathrm{N}^!$ for $1 \leq i \leq m-2$. Since

$$\mathcal{B}(\mathrm{N}^!_{n+1})(i) = y_i\mathcal{B}(\mathrm{N}^!_n)(1) \coprod \cdots \coprod y_i\mathcal{B}(\mathrm{N}^!_n)(i) \coprod y_i\mathcal{B}(\mathrm{N}^!_n)(i+2) \coprod \cdots \coprod y_i\mathcal{B}(\mathrm{N}^!_n)(m-1)$$

for $1 \leq i \leq m-1$ and $n \geq 1$, we can verify the statement. □

Corollary 4.14. *For $m \geq 2$, we have $\varphi(0) = 1, \varphi(1) = m-1$, and $\varphi(2) = m^2 - 3m + 3$. In particular, $\varphi(n) > 0$ for $n \geq 0$.*

Proof. By Proposition 4.13, we can calculate $\varphi(n)$ for $0 \le n \le 2$. (We can also calculate them directly.) We also easily see that $\psi_i(n) > 0$ for $1 \le i \le m-1$ and $n > 0$ by induction. Hence, $\varphi(n) = \sum_{i=1}^{m-1} \psi_i(n) > 0$ for $n > 0$. □

Let N_d be the degree d component of N. Let us define

$$f(t) = \sum_{n=0}^{\infty} (\mathrm{rank}_R \mathrm{N}_n)\, t^n. \tag{4.5}$$

Note that

$$f(t) = 1 + \sum_{k=1}^{m-1} (m-k)t^k = 1 + \sum_{k=1}^{m-1} kt^{m-k}. \tag{4.6}$$

By [11, Chapter 2 Corollary 4.3], the quadratic monomial algebra $\mathrm{N}_m(R)$ over R is Koszul when R is a field. Hence, we have the following formula.

Proposition 4.15. *For $m \ge 2$,*

$$f^!(t)f(-t) = 1.$$

In particular,

$$f^!(t) = \frac{1}{f(-t)} = \frac{1}{1 + \sum_{k=1}^{m-1} (-1)^k (m-k)t^k}.$$

Proof. By [11, Chapter 2 Corollary 2.2] or [7, Theorem 3.5.1], we see that $f^!(t)f(-t) = 1$ if R is a field. The formula can be also proved for any commutative ring R since $f(t)$ and $f^!(t)$ are common to any R. □

The following lemma will be used in §6.

Lemma 4.16. *For $q > 0$, we have*

$$\varphi(q) = \sum_{r=1}^{m-1} (-1)^{r-1}(m-r)\varphi(q-r).$$

Proof. Since $f^!(x)f(-x) = 1$ by Proposition 4.15, we have

$$\left(\sum_{q\ge 0} \varphi(q)x^q\right) \cdot \left(1 + \sum_{k=1}^{m-1} (m-k)(-x)^k\right) = 1.$$

Comparing the coefficients of x^q in both sides, we obtain

$$\varphi(q) - (m-1)\varphi(q-1) + (m-2)\varphi(q-2) - \cdots + (-1)^{m-1}\varphi(q-(m-1)) = 0.$$

This completes the proof. □

We have another formula for $\varphi(n)$.

Proposition 4.17. *For $n \ge 0$, we have*

$$\varphi(n) = (-1)^n \sum_{r\ge 0} (-1)^r \sum_{(a_1,\dots,a_r)} (m-a_1)\cdots(m-a_r), \tag{4.7}$$

where the second sum ranges over the r-tuples $(a_1, \dots, a_r)$ of integers such that $1 \le a_i < m$ for $1 \le i \le r$ and $a_1 + \cdots + a_r = n$.

Proof. Recall $\overline{B}_r(R,\mathrm{N},R)_d = \oplus_{a_1+\cdots+a_r=d}\ \overline{\mathrm{N}}_{a_1}\otimes_R\cdots\otimes_R\overline{\mathrm{N}}_{a_r}$. Since $\mathrm{rank}_R\overline{\mathrm{N}}_a = m-a$ for $1\le a\le m$ and $\overline{\mathrm{N}}_0=0$,

$$\mathrm{rank}_R\overline{B}_r(R,\mathrm{N},R)_d = \sum(m-a_1)\cdots(m-a_r),$$

where the sum ranges over the r-tuples $(a_1,\ldots,a_r)$ of integers such that $1\le a_i<m$ for $1\le i\le r$ and $a_1+\cdots+a_r=d$. The rank of $\overline{C}^{r,d}(\mathrm{N},R)=\mathrm{Hom}_R(\overline{B}_r(R,\mathrm{N},R)_d,R)$ is equal to $\mathrm{rank}_R\overline{B}_r(R,\mathrm{N},R)_d$. Note that the cochain complex

$$0\longrightarrow\overline{C}^{0,d}(\mathrm{N},R)\longrightarrow\overline{C}^{1,d}(\mathrm{N},R)\longrightarrow\overline{C}^{2,d}(\mathrm{N},R)\longrightarrow\cdots \tag{4.8}$$

satisfies that $\overline{C}^{r,d}(\mathrm{N},R)=0$ for $r>d$. By taking the Euler characteristic of (4.8) and using $\mathrm{HH}^{r,d}(\mathrm{N},R)\cong H^r(\overline{C}^{*,d}(\mathrm{N},R))$, we obtain

$$\sum_{r\ge0}(-1)^r\mathrm{rank}_R\mathrm{HH}^{r,d}(\mathrm{N},R)=\sum_{r\ge0}(-1)^r\mathrm{rank}_R\overline{C}^{r,d}(\mathrm{N},R).$$

By Theorem 4.10, $\mathrm{HH}^{r,d}(\mathrm{N},R)=0$ for $r\ne d$ and $\mathrm{HH}^{d,d}(\mathrm{N},R)\cong\mathrm{N}^!_d$. Hence,

$$(-1)^d\mathrm{rank}_R\mathrm{N}^!_d=\sum_{r\ge0}(-1)^r\mathrm{rank}_R\overline{C}^{r,d}(\mathrm{N},R).$$

Thus, we have

$$\begin{aligned}\varphi(n) &= \mathrm{rank}_R\mathrm{N}^!_n\\ &= (-1)^n\sum_{r\ge0}(-1)^r\mathrm{rank}_R\overline{C}^{r,n}(\mathrm{N},R)\\ &= (-1)^n\sum_{r\ge0}(-1)^r\sum_{(a_1,\ldots,a_r)}(m-a_1)\cdots(m-a_r),\end{aligned}$$

which is what we wanted to show. □

By taking account of the appearance of the term $(m-1)^{i_1}(m-2)^{i_2}\cdots(m-(m-1))^{i_{m-1}}$ in the right hand side of (4.7) in Proposition 4.17, we obtain the following corollary.

Corollary 4.18. *For $n\ge0$, we have*

$$\varphi(n)=(-1)^n\sum\frac{(i_1+i_2+\cdots+i_{m-1})!}{i_1!i_2!\cdots i_{m-1}!}(1-m)^{i_1}(2-m)^{i_2}\cdots((m-1)-m)^{i_{m-1}}, \tag{4.9}$$

where the sum ranges over the $(m-1)$-tuples $(i_1,i_2,\ldots,i_{m-1})$ of non-negative integers such that $i_1+2i_2+\cdots+(m-1)i_{m-1}=n$.

Remark 4.19. Corollary 4.18 can be also proved by using Proposition 4.15. Indeed, putting $y=-\sum_{k=1}^{m-1}(-1)^k(m-k)t^k=\sum_{k=1}^{m-1}(k-m)(-t)^k$, we have

$$f^!(t)=\frac{1}{f(-t)}=\frac{1}{1-y}=1+y+y^2+y^3+\cdots$$

by Proposition 4.15. Then $f^!(t)$ equals to

$$\begin{aligned}\sum_{r\ge0}y^r &= \sum_{r\ge0}\left\{\sum_{k=1}^{m-1}(k-m)(-t)^k\right\}^r\\ &= \sum_{n\ge0}(-1)^nt^n\sum\frac{(i_1+i_2+\cdots+i_{m-1})!}{i_1!i_2!\cdots i_{m-1}!}(1-m)^{i_1}(2-m)^{i_2}\cdots((m-1)-m)^{i_{m-1}},\end{aligned}$$

where the second sum ranges over the $(m-1)$-tuples $(i_1, i_2, \ldots, i_{m-1})$ of non-negative integers such that $i_1 + 2i_2 + \cdots + (m-1)i_{m-1} = n$. Comparing the coefficients of t^n on both sides, we obtain (4.9).

5. $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$

In this section, we determine the R-module structure of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ for $m \geq 3$. (The case $m = 2$ will be discussed in §9.) In §5.1, we construct a spectral sequence converging to $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$, which collapses from the E_2-page. In §5.2, we show that $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ is a free R-module by calculating $E_2^{p,q}$. We also calculate the rank of the free R-module $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ by using $\varphi(n)$ defined in Definition 4.11.

5.1. **Spectral sequences for subquotients of $\mathrm{M}_m(R)$.** In this subsection we introduce a $\mathbb{Z}$-grading on the matrix algebra $\mathrm{M}_m(R)$. Using this grading, we construct a spectral sequence of $\mathbb{Z}$-graded R-modules converging to the Hochschild cohomology $\mathrm{HH}^*(\mathrm{N}_m(R), \widehat{M})$, where $\widehat{M}$ is a subquotient of the $\mathbb{Z}$-graded R-module $\mathrm{M}_m(R)$. Furthermore, we show that this spectral sequence collapses from the E_2-page.

In this subsection we work in the abelian category of $\mathbb{Z}$-graded R-modules. Let $m \geq 2$ and $\mathrm{M} = \mathrm{M}_m(R)$. First, we introduce a grading on M. We can choose a basis

$$\{E_{i,j} |\ 1 \leq i, j \leq m\}$$

over R. We define a (homological) degree r component of M by

$$\mathrm{M}_r = \bigoplus_{j-i=r} R\{E_{i,j}\}.$$

Then we can verify that $\mathrm{M} = \bigoplus_{r\in\mathbb{Z}} \mathrm{M}_r$ is a $\mathbb{Z}$-graded associative algebra over R.

Let $\mathrm{N} = \mathrm{N}_m(R)$. We can easily see that N is a $\mathbb{Z}$-graded subalgebra of M. For a $\mathbb{Z}$-graded N-bimodule L over R, we let $C^*(\mathrm{N}, L)$ be the Hochschild cochain complex. We have

$$\begin{aligned} C^p(\mathrm{N}, L) &\cong \mathrm{Hom}_R(\mathrm{N}^{\otimes p}, L) \\ &\cong (\mathrm{N}^*)^{\otimes p} \otimes_R L, \end{aligned}$$

where $\mathrm{N}^* = \mathrm{Hom}_R(\mathrm{N}, R)$. We denote by

$$C^{p,s}(\mathrm{N}, L)$$

the R-submodule of $C^p(\mathrm{N}, L)$ of (cohomological) degree s. For example, when $L = \mathrm{N}$, we have

$$E_{1,2} \in C^{0,-1}(\mathrm{N}, \mathrm{N}), \quad E_{2,3}^* \otimes I_m \in C^{1,1}(\mathrm{N}, \mathrm{N}), \quad E_{1,2}^* \otimes E_{1,3}^* \otimes E_{2,3} \in C^{2,2}(\mathrm{N}, \mathrm{N}),$$

where $\{I_m^*\} \cup \{E_{i,j}^* \mid i < j\}$ is the dual basis of N^* with respect to the R-basis $\{I_m\} \cup \{E_{i,j} \mid i < j\}$ of N.

Since the differential $d : C^p(\mathrm{N}, L) \to C^{p+1}(\mathrm{N}, L)$ preserves the grading, we have an isomorphism

$$C^*(\mathrm{N}, L) \cong \bigoplus_{s\in\mathbb{Z}} C^{*,s}(\mathrm{N}, L)$$

of cochain complexes of R-modules. Thus, we can regard $C^*(\mathrm{N}, L)$ as a cochain complex of $\mathbb{Z}$-graded R-modules. We set

$$\mathrm{HH}^{n,s}(\mathrm{N}, L) = H^n(C^{*,s}(\mathrm{N}, L)) \tag{5.1}$$

as in Notation 3.7.

Let

$$\mathrm{J} = \bigoplus_{j-i>0} R\{E_{i,j}\}$$

be the two-sided ideal of N consisting of upper triangular matrices with zero diagonal entries. In other words, J coincides with $J(\mathrm{N}_m(R))$ defined in §4.2. By (3.1), we have the filtration $\{\overline{\mathrm{J}}^p L\}_{p\geq 0}$ on M, which induces a filtration on the cochain complex $C^*(\mathrm{N}, L)$. Since J is a homogeneous two-sided ideal of N, we can verify that this filtration is compatible with the grading. Thus, we obtain a spectral sequence

$$ {}^J E_1^{p,q}(\mathrm{N}, L) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, L)$$

of $\mathbb{Z}$-graded R-modules by Proposition 3.6, where

$$ {}^J E_1^{p,q}(\mathrm{N}, L) \cong \mathrm{HH}^{p+q}(\mathrm{N}, \overline{\mathrm{J}}^p L/\overline{\mathrm{J}}^{p+1} L).$$

We note that

$$ {}^J E_r^{p,q}(\mathrm{N}, L) = 0$$

for $p \geq 2m-1$ since $\overline{\mathrm{J}}^{2m-1} L = 0$.

Next, we consider the following situation. Let $M'' \subset M' \subset \mathrm{M} = \mathrm{M}_m(R)$ be $\mathbb{Z}$-graded N-sub-bimodules over R. We would like to construct another spectral sequence converging to the Hochschild cohomology $\mathrm{HH}^*(\mathrm{N}, \widehat{M})$, where $\widehat{M} = M'/M''$.

For this purpose, we define a filtration $\{F^p\mathrm{M}\}$ on the $\mathbb{Z}$-graded R-module $\mathrm{M} = \mathrm{M}_m(R)$ by reindexing the filtration $\{\overline{\mathrm{J}}^p\mathrm{M}\}$ as follows

$$F^{p-(m-1)}\mathrm{M} = \overline{\mathrm{J}}^p\mathrm{M} = \sum_{a+b=p} \mathrm{J}^a\mathrm{M}\mathrm{J}^b.$$

Then we have

$$\mathrm{M} = F^{-(m-1)}\mathrm{M} \supset F^{-(m-1)+1}\mathrm{M} \supset \cdots \supset F^{m-1}\mathrm{M} \supset F^m\mathrm{M} = 0.$$

Note that

$$\mathrm{Gr}^p(\mathrm{M}) \cong R\{E_{i,j} |\ j-i = p\}.$$

Using this filtration on M, we define a filtration $\{F^p\widehat{M}\}$ on $\widehat{M}$ to be the induced filtration

$$F^p\widehat{M} = ((M' \cap F^p\mathrm{M}) + M'')/M''.$$

Note that $\mathrm{Gr}^p(\widehat{M}) = F^p\widehat{M}/F^{p+1}\widehat{M}$ is a subquotient of $\mathrm{Gr}^p(\mathrm{M})$. Using this filtration on $\widehat{M}$, we obtain the following proposition.

Proposition 5.1. *There is a spectral sequence*

$$ {}^M E_1^{p,q}(\mathrm{N}, \widehat{M}) \Longrightarrow H^{p+q}(\mathrm{N}, \widehat{M})$$

of $\mathbb{Z}$-graded R-modules, where

$$ {}^M E_1^{p,q}(\mathrm{N}, \widehat{M}) \cong \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{Gr}^p(\widehat{M})).$$

We have ${}^M E_r^{p,q}(\mathrm{N}, \widehat{M}) = 0$ unless $-(m-1) \leq p \leq m-1$.

We shall show that the spectral sequence $\{{}^M E_r^{p,q}(\mathrm{N}, \widehat{M})\}_{r\geq 1}$ collapses from the E_2-page and that there is no extension problem. Recall the degree s component ${}^M E_r^{p,q,s}(\mathrm{N}, \widehat{M})$ of $\{{}^M E_r^{p,q}(\mathrm{N}, \widehat{M})\}_{r\geq 1}$ in Proposition 3.6.

Lemma 5.2. *We have*

$$ {}^M E_1^{p,q,s}(\mathrm{N}, \widehat{M}) \cong \begin{cases} \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{Gr}^p(\widehat{M})) & (s = q), \\ 0 & (s \neq q). \end{cases}$$

Proof. Since $\mathrm{J}\cdot \mathrm{Gr}^p(\widehat{M}) = \mathrm{Gr}^p(\widehat{M})\cdot \mathrm{J} = 0$ for each $p \geq 0$, we have an isomorphism

$$ {}^{M}E_1^{p,q}(\mathrm{N}, \widehat{M}) \cong \mathrm{HH}^{p+q}(\mathrm{N}, R) \otimes_R \mathrm{Gr}^p(\widehat{M}) $$

by Proposition 4.9. Recall that

$$ \mathrm{HH}^{n,d}(\mathrm{N}, R) \cong \begin{cases} \mathrm{N}^!_n & (d = n), \\ 0 & (d \neq n) \end{cases} $$

by Theorem 4.10. The statement follows from the fact that $\mathrm{Gr}^p(\widehat{M})$ is a subquotient of $\mathrm{Gr}^p(\mathrm{M}) = R\{E_{i,j} \mid j - i = p\}$. □

Theorem 5.3. *The spectral sequence* ${}^{M}E_1^{p,q}(\mathrm{N}, \widehat{M}) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \widehat{M})$ *collapses from the* E_2*-page and there is no extension problem.*

Proof. By Lemmas 3.8 and 5.2, we see that the spectral sequence collapses from the E_2-page.

We shall show that there is no extension problem. We have a filtration $\{F^p\mathrm{HH}^n(\mathrm{N}, \widehat{M})\}$ on the $\mathbb{Z}$-graded R-module $\mathrm{HH}^n(\mathrm{N}, \widehat{M})$ given by

$$ F^p\mathrm{HH}^n(\mathrm{N}, \widehat{M}) = \mathrm{Im}(\mathrm{HH}^n(\mathrm{N}, F^p\widehat{M}) \longrightarrow \mathrm{HH}^n(\mathrm{N}, \widehat{M})). $$

By Lemma 5.2, ${}^{M}E_\infty^{p,q,s} = 0$ for $s \neq q$. This implies that the exact sequence

$$ 0 \longrightarrow F^{p+1}\mathrm{HH}^n(\mathrm{N}, \widehat{M}) \longrightarrow F^p\mathrm{HH}^n(\mathrm{N}, \widehat{M}) \longrightarrow {}^{M}E_\infty^{p,n-p}(\mathrm{N}, \widehat{M}) \longrightarrow 0 $$

is canonically split. We obtain a canonical isomorphism

$$ \mathrm{HH}^n(\mathrm{N}; \widehat{M}) \cong \bigoplus_p {}^{M}E_\infty^{p,n-p}(\mathrm{N}, \widehat{M}) $$

and hence there is no extension problem. □

In particular, applying Theorem 5.3 to the case where $\widehat{M} = \mathrm{M}/\mathrm{N}$, we obtain the following corollary.

Corollary 5.4. *The spectral sequence*

$$ {}^{M}E_1^{p,q}(\mathrm{N}, \mathrm{M}/\mathrm{N}) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{M}/\mathrm{N}) $$

of $\mathbb{Z}$*-graded* R*-modules collapses from the* E_2*-page. There is an isomorphism*

$$ \mathrm{HH}^n(\mathrm{N}, \mathrm{M}/\mathrm{N}) = \bigoplus_{n,s} \mathrm{HH}^{n,s}(\mathrm{N}, \mathrm{M}/\mathrm{N}) \cong \bigoplus_{n,s} {}^{M}E_\infty^{n-s,s}(\mathrm{N}, \mathrm{M}/\mathrm{N}) $$

of bigraded R*-modules.*

5.2. **Calculation of** $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$**.** In this subsection, we assume that $m \geq 3$. Let us calculate $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$. By Corollary 5.4, we only need to calculate $E_2^{p,q}(\mathrm{N}, \mathrm{M}/\mathrm{N})$. Put $F^p = F^p(\mathrm{M}/\mathrm{N})$ and $E_r^{p,q} = E_r^{p,q}(\mathrm{N}, \mathrm{M}/\mathrm{N})$. Recall that

$$ \mathrm{M}/\mathrm{N} = F^{-(m-1)} \supset F^{-(m-2)} \supset F^{-(m-3)} \supset \cdots \supset F^0 = \mathrm{B}/\mathrm{N} \supset F^1 = 0, $$

where $\mathrm{B} = \mathrm{B}_m(R) = \{(a_{ij}) \in \mathrm{M}_m(R) \mid a_{ij} = 0 \text{ for } i > j\}$. It is easy to see that $\mathrm{Gr}^p(\mathrm{M}/\mathrm{N}) = F^p/F^{p+1}$ is isomorphic to the direct sum of finitely many copies of R as an N-bimodule over R. Hence we have

$$ E_1^{p,q} \cong \mathrm{HH}^{p+q}(\mathrm{N}, R) \otimes_R (F^p/F^{p+1}) \cong \mathrm{N}^!_{p+q} \otimes_R (F^p/F^{p+1}). $$

Since F^p/F^{p+1} is a free module over R, so is $E_1^{p,q}$. Note that $\mathrm{rank}_R(F^{-(m-1)}/F^{-(m-2)}) = 1$, $\mathrm{rank}_R(F^{-(m-2)}/F^{-(m-3)}) = 2$, $\ldots$, $\mathrm{rank}_R(F^{-1}/F^0) = m-1$ and $\mathrm{rank}_R(F^0/F^1) = m-1$. Then we have

$$\begin{array}{rcl} \mathrm{rank}_R E_1^{-(m-1),q} & = & \varphi(q-m+1), \\ \mathrm{rank}_R E_1^{-(m-2),q} & = & 2\varphi(q-m+2), \\ \mathrm{rank}_R E_1^{-(m-3),q} & = & 3\varphi(q-m+3), \\ \cdots & \cdots & \cdots \\ \mathrm{rank}_R E_1^{-1,q} & = & (m-1)\varphi(q-1), \\ \mathrm{rank}_R E_1^{0,q} & = & (m-1)\varphi(q), \end{array}$$

where $\varphi(n) = \mathrm{rank}_R \mathrm{N}^!_n$ in Definition 4.11. The R-homomorphism $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ can be identified with the connecting homomorphism $\delta : \mathrm{HH}^{p+q}(\mathrm{N}, F^p/F^{p+1}) \to \mathrm{HH}^{p+q+1}(\mathrm{N}, F^{p+1}/F^{p+2})$ obtained by the short exact sequence $0 \to F^{p+1}/F^{p+2} \to F^p/F^{p+2} \to F^p/F^{p+1} \to 0$.

We can write

$$\mathrm{Gr}^p(\mathrm{M/N}) = F^p/F^{p+1} = \begin{cases} RE_{1-p,1} \oplus RE_{2-p,2} \oplus \cdots \oplus RE_{m,m+p} & (-(m-1) \le p \le -1), \\ (RE_{1,1} \oplus RE_{2,2} \oplus \cdots RE_{m,m})/RI_m & (p=0). \end{cases}$$

By Proposition 4.1 (3), $\widehat{K}_*(A)$ gives us a graded free resolution of A as A-bimodules over R if $A = \mathrm{N}$. Recall the differential $\widehat{d}_n : A \otimes_R (A^!_n)^* \otimes_R A \to A \otimes_R (A^!_{n-1})^* \otimes_R A$ is given by (4.1). For $-(m-1) \le p \le -1$,

$$\begin{array}{rccl} d_1^{p,q}: & \mathrm{N}^!_{p+q} \otimes \mathrm{Gr}^p(\mathrm{M/N}) & \longrightarrow & \mathrm{N}^!_{p+q+1} \otimes \mathrm{Gr}^{p+1}(\mathrm{M/N}) \\ & f \otimes E_{i-p,i} & \longmapsto & y_{i-p-1} f \otimes E_{i-p-1,i} + (-1)^{p+q+1} f y_i \otimes E_{i-p,i+1} \end{array}$$

for $1 \le i \le m+p$.

Let us consider the complex

$$0 \longrightarrow E_1^{-(m-1),q} \overset{d_1^{-(m-1),q}}{\longrightarrow} E_1^{-(m-2),q} \overset{d_1^{-(m-2),q}}{\longrightarrow} E_1^{-(m-3),q} \overset{d_1^{-(m-3),q}}{\longrightarrow} \cdots \overset{d_1^{-2,q}}{\longrightarrow} E_1^{-1,q} \overset{d_1^{-1,q}}{\longrightarrow} E_1^{0,q} \longrightarrow 0.$$

We define another complex

$$0 \longrightarrow C_1^{-(m-1),q} \overset{\delta_1^{-(m-1),q}}{\longrightarrow} C_1^{-(m-2),q} \overset{\delta_1^{-(m-2),q}}{\longrightarrow} C_1^{-(m-3),q} \overset{\delta_1^{-(m-3),q}}{\longrightarrow} \cdots \overset{\delta_1^{-2,q}}{\longrightarrow} C_1^{-1,q} \overset{\delta_1^{-1,q}}{\longrightarrow} C_1^{0,q} \longrightarrow 0$$

by $C^{i,q} = E_1^{i,q}$ for $-(m-1) \le i \le -1$, $C^{0,q} = \mathrm{N}^!_q \otimes_R (RE_{1,1} \oplus RE_{2,2} \oplus \cdots \oplus RE_{m,m})$, $\delta^{i,q} = d_1^{i,q}$ for $-(m-1) \le i \le -2$, and

$$\begin{array}{rccl} \delta^{-1,q}: & C^{-1,q} = \mathrm{N}^!_{q-1} \otimes \mathrm{Gr}^{-1}(\mathrm{M/N}) & \longrightarrow & C^{0,q} = \mathrm{N}^!_q \otimes_R (\oplus_{k=1}^m RE_{k,k}) \\ & f \otimes E_{i+1,i} & \longmapsto & y_i f \otimes E_{i,i} + (-1)^q f y_i \otimes E_{i+1,i+1} \end{array}$$

for $1 \le i \le m-1$. Then we have a homomorphism of complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C^{-(m-1),q} & \overset{\delta^{-(m-1),q}}{\longrightarrow} & C^{-(m-2),q} & \overset{\delta^{-(m-2),q}}{\longrightarrow} & \cdots & \overset{\delta^{-2,q}}{\longrightarrow} & C^{-1,q} & \overset{\delta^{-1,q}}{\longrightarrow} & C^{0,q} & \longrightarrow & 0 \\ & & \downarrow \phi_{-(m-1)} & & \downarrow \phi_{-(m-2)} & & & & \downarrow \phi_{-1} & & \downarrow \phi_0 & & \\ 0 & \longrightarrow & E_1^{-(m-1),q} & \overset{d_1^{-(m-1),q}}{\longrightarrow} & E_1^{-(m-2),q} & \overset{d_1^{-(m-2),q}}{\longrightarrow} & \cdots & \overset{d_1^{-2,q}}{\longrightarrow} & E_1^{-1,q} & \overset{d_1^{-1,q}}{\longrightarrow} & E_1^{0,q} & \longrightarrow & 0, \end{array}$$

where $\phi_i = id_{E_1^{i,q}}$ for $-(m-1) \le i \le -1$ and $\phi_0 : C^{0,q} = \mathrm{N}^!_q \otimes_R (\oplus_{k=1}^m RE_{k,k}) \to E_1^{0,q} = \mathrm{N}^!_q \otimes_R ((\oplus_{k=1}^m RE_{k,k})/RI_m)$ is the projection.

Remark 5.5. We can give an interpretation of $C^{p,q}$ from a viewpoint of spectral sequences. Let $\mathrm{J} = J(\mathrm{N}_m(R))$. As in the case M/N, we put $F'^p = F^p(\mathrm{M/J})$. We have a filtration

$$\mathrm{M/J} = F'^{-(m-1)} \supset F'^{-(m-2)} \supset \cdots \supset F'^1 \supset F'^0 \supset F'^{-1} = 0$$

of A-bimodules over R. We denote by $\mathrm{Gr}^p(\mathrm{M/J})$ the p-th associated graded module F'^p/F'^{p+1}. By Proposition 5.1, we obtain a spectral sequence

$$E_1'^{p,q} = \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{Gr}^p(\mathrm{M/J})) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{M/J})$$

with

$$d_r : E_r'^{p,q} \longrightarrow E_r'^{p+r,q-r+1}$$

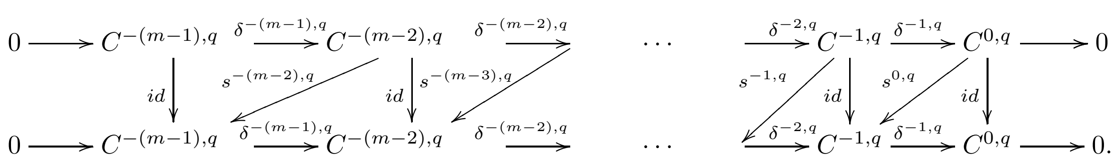
for $r \geq 1$. Then $\{C^{p,q}\} \cong \{E_1'^{p,q}\}$ as chain complexes. The homomorphism $\{C^{p,q} \to E_1^{p,q}\}$ of chain complexes can be identified with the canonical map of spectral sequences $E_1'^{p,q} \to E_1^{p,q}$.

Set $G^p = \mathrm{Gr}^p(\mathrm{M/N})$ for $-(m-1) \leq p \leq -1$ and $G^0 = \oplus_{k=1}^m RE_{k,k}$. For $-(m-1) \leq p \leq -1$, we define the R-homomorphism $s^{p+1,q} : C^{p+1,q} \to C^{p,q}$ by

$$\begin{array}{ccc} C^{p+1,q} = \mathrm{N}^!_{p+q+1} \otimes G^{p+1} & \overset{s^{p+1,q}}{\longrightarrow} & C^{p,q} = \mathrm{N}^!_{p+q} \otimes G^p \\ f \otimes E_{i,i+p+1} & \longmapsto & \begin{cases} 0 & \text{if } f \notin y_i\mathcal{B}(\mathrm{N}^!_{p+q}) \\ f' \otimes E_{i+1,i+p+1} & \begin{array}{l}\text{if } f = y_i f' \\ \text{for } f' \in \mathcal{B}(\mathrm{N}^!_{p+q})\end{array} \end{cases} \quad (-p \leq i \leq m-1), \\ f \otimes E_{m,m+p+1} & \longmapsto & 0 \qquad (i = m) \end{array}$$

for $f \in \mathcal{B}(\mathrm{N}^!_{p+q+1})$.

Lemma 5.6. *For $-(m-2) \leq p \leq -1$, $\delta^{p-1,q} \circ s^{p,q} + s^{p+1,q} \circ \delta^{p,q} = id_{C^{p,q}}$ and $s^{-(m-2),q} \circ \delta^{-(m-1),q} = id_{C^{-(m-1),q}}$:*



Proof. Let us prove that $s^{-(m-2),q} \circ \delta^{-(m-1),q} = id_{C^{-(m-1),q}}$. For $f \otimes E_{m,1} \in C^{-(m-1),q} = \mathrm{N}^!_{q-(m-1)} \otimes_R RE_{m,1}$,

$$\begin{aligned} s^{-(m-2),q} \circ \delta^{-(m-1),q}(f \otimes E_{m,1}) &= s^{-(m-2),q}(y_{m-1}f \otimes E_{m-1,1} + (-1)^{q-m+2} f y_1 \otimes E_{m,2}) \\ &= f \otimes E_{m,1}. \end{aligned}$$

Hence, $s^{-(m-2),q} \circ \delta^{-(m-1),q} = id_{C^{-(m-1),q}}$.

Let us show that $\delta^{p-1,q} \circ s^{p,q} + s^{p+1,q} \circ \delta^{p,q} = id_{C^{p,q}}$ for $-(m-2) \leq p \leq -1$. Note that $C^{p,q} = \mathrm{N}^!_{p+q} \otimes_R (RE_{1-p,1} \oplus RE_{2-p,2} \oplus \cdots \oplus RE_{m,m+p})$. It suffices to prove that $(\delta^{p-1,q} \circ s^{p,q} + s^{p+1,q} \circ \delta^{p,q})(f \otimes E_{i-p,i}) = f \otimes E_{i-p,i}$ for $-(m-2) \leq p \leq -1$, $1 \leq i \leq m+p$, and $f \in \mathcal{B}(\mathrm{N}^!_{p+q})$. Note that

$$\delta^{p,q}(f \otimes E_{i-p,i}) = y_{i-p-1} f \otimes E_{i-p-1,i} + (-1)^{p+q+1} f y_i \otimes E_{i-p,i+1}.$$

Assume that $1 \leq i \leq m+p-1$ and that $f \in \mathcal{B}(\mathrm{N}^!_{p+q})$. Then

$$s^{p+1,q} \circ \delta^{p,q}(f \otimes E_{i-p,i}) = \begin{cases} f \otimes E_{i-p,i} & (f \notin y_{i-p}\mathcal{B}(\mathrm{N}^!_{p+q-1})), \\ (-1)^{p+q+1} f' y_i \otimes E_{i-p+1,i+1} & (f = y_{i-p} f'). \end{cases}$$

Since

$$s^{p,q}(f \otimes E_{i-p,i}) = \begin{cases} 0 & (f \notin y_{i-p}\mathcal{B}(\mathrm{N}^!_{p+q-1})), \\ f' \otimes E_{i-p+1,i} & (f = y_{i-p} f'), \end{cases}$$

we have

$$\begin{aligned}&\delta^{p-1,q}\circ s^{p,q}(f\otimes E_{i-p,i})\\ =&\left\{\begin{array}{cc} 0 & (f\notin y_{i-p}\mathcal{B}(\mathrm{N}^!_{p+q-1})),\\ f\otimes E_{i-p,i}+(-1)^{p+q}f'y_iE_{i-p+1,i+1} & (f=y_{i-p}f').\end{array}\right.\end{aligned}$$

Thereby, we obtain

$$(\delta^{p-1,q}\circ s^{p,q}+s^{p+1,q}\circ\delta^{p,q})(f\otimes E_{i-p,i})=f\otimes E_{i-p,i}.$$

Assume that $i=m+p$ and that $f\in\mathcal{B}(\mathrm{N}^!_{p+q})$. Then $s^{p+1,q}\circ\delta^{p,q}(f\otimes E_{m,m+p})=f\otimes E_{m,m+p}$. Since $s^{p,q}(f\otimes E_{m,m+p})=0$, $\delta^{p-1,q}\circ s^{p,q}(f\otimes E_{m,m+p})=0$. Hence $(\delta^{p-1,q}\circ s^{p,q}+s^{p+1,q}\circ\delta^{p,q})(f\otimes E_{m,m+p})=f\otimes E_{m,m+p}$. This completes the proof. □

Lemma 5.7. *Let* $K^{0,q}=\mathrm{N}^!_q\otimes_R RI_m\subseteq C^{0,q}=\mathrm{N}^!_q\otimes_R(\sum_{k=1}^m RE_{k,k})$. *Then* $\delta^{-1,q}(C^{-1,q})\cap K^{0,q}=0$.

Proof. Let $x=f_2\otimes E_{2,1}+f_3\otimes E_{3,2}+\cdots+f_i\otimes E_{i,i-1}+\cdots+f_m\otimes E_{m,m-1}\in C^{-1,q}$, where $f_i\in\mathrm{N}^!_{q-1}$ $(2\le i\le m)$. Note that

$$\begin{aligned}\delta^{-1,q}(x)=\;&y_1f_2\otimes E_{1,1}+((-1)^qf_2y_1+y_2f_3)E_{2,2}+((-1)^qf_3y_2+y_3f_4)E_{3,3}+\cdots\\&+((-1)^qf_iy_{i-1}+y_if_{i+1})E_{i,i}+\cdots+((-1)^qf_{m-1}y_{m-2}+y_{m-1}f_m)E_{m-1,m-1}\\&+(-1)^qf_my_{m-1}\otimes E_{m,m}.\end{aligned}$$

Suppose that $\delta^{-1,q}(x)\in K^{0,q}$. Then

$$y_1f_2=(-1)^qf_2y_1+y_2f_3, \tag{5.2}$$

$$(-1)^qf_2y_1+y_2f_3=(-1)^qf_3y_2+y_3f_4, \tag{5.3}$$

$$\cdots$$

$$(-1)^qf_{m-1}y_{m-2}+y_{m-1}f_m=(-1)^qf_my_{m-1}. \tag{5.m}$$

We can write $f_2=f_2'+y_2f_2''$ such that $f_2'\in y_1\mathrm{N}^!_{q-2}\oplus y_3\mathrm{N}^!_{q-2}\oplus\cdots\oplus y_{m-1}\mathrm{N}^!_{q-2}$ and $f_2''\in\mathrm{N}^!_{q-2}$. By (5.2), we obtain

$$y_1f_2'\;=\;(-1)^qf_2'y_1+(-1)^qy_2f_2''y_1+y_2f_3,$$

and hence

$$y_1f_2'-(-1)^qf_2'y_1\;=\;y_2((-1)^qf_2''y_1+f_3). \tag{5.4}$$

Since the right hand side of (5.4) has the leading term y_2, we have $y_1f_2'-(-1)^qf_2'y_1=0$. Hence $y_1f_2'=(-1)^qf_2'y_1$. We see that $f_2'=cy_1^{q-1}$ for some $c\in R$.

Similarly, we can write $f_m=f_m'+f_m''y_{m-2}$ such that $f_m'\in\mathrm{N}^!_{q-2}y_1\oplus\cdots\oplus\mathrm{N}^!_{q-2}y_{m-3}\oplus\mathrm{N}^!_{q-2}y_{m-1}$ and $f_m''\in\mathrm{N}^!_{q-2}$. By (5.$m$), we obtain

$$(-1)^qf_{m-1}y_{m-2}+y_{m-1}f_m'+y_{m-1}f_m''y_{m-2}\;=\;(-1)^qf_m'y_{m-1},$$

and hence

$$((-1)^qf_{m-1}+y_{m-1}f_m'')y_{m-2}\;=\;(-1)^qf_m'y_{m-1}-y_{m-1}f_m'. \tag{5.5}$$

Since the left hand side of (5.5) has the last term y_{m-2}, we have $(-1)^qf_m'y_{m-1}-y_{m-1}f_m'=0$. Hence $y_{m-1}f_m'=(-1)^qf_m'y_{m-1}$. We see that $f_m'=dy_{m-1}^{q-1}$ for some $d\in R$.

By using (5.2), (5.3), …, and (5.m), $y_1f_2=(-1)^qf_my_{m-1}$. Then we obtain

$$\begin{aligned}y_1(f_2'+y_2f_2'')\;&=\;(-1)^q(f_m'+f_m''y_{m-2})y_{m-1},\\ y_1f_2'\;&=\;(-1)^qf_m'y_{m-1}.\end{aligned}$$

Since $f_2' = cy_1^{q-1}$ and $f_m' = dy_{m-1}^{q-1}$, $cy_1^q = (-1)^q dy_{m-1}^q$, which implies that $c = d = 0$. Hence $y_1 f_2 = y_1(f_2' + y_2 f_2'') = y_1 y_2 f_2'' = 0$. Therefore $\delta^{-1,q}(x) = 0$ by (5.2), (5.3), …, and (5.m). This completes the proof. □

Proposition 5.8. *For $-(m-1) \le p \le -1$, we have $E_2^{p,q} = 0$ for all q. In particular, $E_2^{p,q} = 0$ unless $p = 0$.*

Proof. For $-(m-1) \le p \le -2$, $E_2^{p,q} \cong H^p(C^{*,q}) = 0$ by Lemma 5.6. For $p = -1$, let us consider the commutative diagram with columns exact:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & K^{0,q} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \stackrel{\delta^{-3,q}}{\longrightarrow} & C^{-2,q} & \stackrel{\delta^{-2,q}}{\longrightarrow} & C^{-1,q} & \stackrel{\delta^{-1,q}}{\longrightarrow} & C^{0,q} & \longrightarrow & 0 \\
 & & \downarrow \phi_{-2} & & \downarrow \phi_{-1} & & \downarrow \phi_0 & & \\
\cdots & \stackrel{d_1^{-3,q}}{\longrightarrow} & E_1^{-2,q} & \stackrel{d_1^{-2,q}}{\longrightarrow} & E_1^{-1,q} & \stackrel{d_1^{-1,q}}{\longrightarrow} & E_1^{0,q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

It is easy to see that $E_2^{-1,q} \cong H^{-1}(C^{*,q}) = 0$ by Lemmas 5.6 and 5.7. The last statement follows from that $E_1^{p,q} = 0$ unless $-(m-1) \le p \le 0$. □

We introduce the following claim. This is true for the case R is a field.

Claim 5.9. *The R-module $E_2^{0,q} \cong E_1^{0,q}/\mathrm{Im}\, d_1^{-1,q}$ is free.*

Under the hypothesis that Claim 5.9 is true, let us calculate $\mathrm{rank}_R E_2^{0,q}$. By taking the Euler characteristic of the cochain complex

$$0 \longrightarrow E_1^{-(m-1),q} \stackrel{d_1^{-(m-1),q}}{\longrightarrow} E_1^{-(m-2),q} \stackrel{d_1^{-(m-2),q}}{\longrightarrow} \cdots \stackrel{d_1^{-2,q}}{\longrightarrow} E_1^{-1,q} \stackrel{d_1^{-1,q}}{\longrightarrow} E_1^{0,q} \longrightarrow 0,$$

we obtain

$$\begin{aligned}
\sum_{k=-(m-1)}^{0} (-1)^k \mathrm{rank}_R E_2^{k,q} &= \sum_{k=-(m-1)}^{0} (-1)^k \mathrm{rank}_R E_1^{k,q} \\
&= (m-1)\varphi(q) + \sum_{k=-(m-1)}^{-1} (-1)^k (m+k)\varphi(q+k).
\end{aligned}$$

Since $E_2^{k,q} = 0$ for $-(m-1) \le k \le -1$ by Proposition 5.8,

$$\mathrm{rank}_R E_2^{0,q} = (m-1)\varphi(q) + \sum_{k=-(m-1)}^{-1} (-1)^k (m+k)\varphi(q+k).$$

Hence

$$\mathrm{rank}_R E_2^{0,q} = (m-1)\varphi(q) + \sum_{k=1}^{m-1} (-1)^{m+k} k\varphi(q-m+k). \tag{5.6}$$

Now, let us prove Claim 5.9. To emphasize R, we denote by $d_1^{-1,q}(R) : E_1^{-1,q}(R) \to E_1^{0,q}(R)$ the differential $d_1^{-1,q} : E_1^{-1,q} \to E_1^{0,q}$. We can ragard $d_1^{-1,q}(S)$ as $d_1^{-1,q}(R) \otimes_R S$ for any ring homomorphism $R \to S$.

Lemma 5.10. *The R-module $E_2^{0,q} \cong E_1^{0,q}/\mathrm{Im}\ d_1^{-1,q}$ is a free module over R. The rank of $E_2^{0,q}$ over R is given by (5.6).*

Proof. When R is a field, the statement is true. For $R = \mathbb{Z}$, let us consider the exact sequence

$$E_1^{-1,q}(\mathbb{Z}) \overset{d_1^{-1,q}(\mathbb{Z})}{\longrightarrow} E_1^{0,q}(\mathbb{Z}) \longrightarrow E_2^{0,q}(\mathbb{Z}) \longrightarrow 0. \tag{5.7}$$

Note that $E_2^{0,q}(\mathbb{Z})$ is finitely generated over $\mathbb{Z}$. Suppose that $E_2^{0,q}(\mathbb{Z})$ has a torsion element $x \neq 0$ such that $px = 0$ for a prime number p. By tensoring (5.7) by $\mathbb{F}_p$, we have an exact sequence

$$E_1^{-1,q}(\mathbb{F}_p) \overset{d_1^{-1,q}(\mathbb{F}_p)}{\longrightarrow} E_1^{0,q}(\mathbb{F}_p) \longrightarrow E_2^{0,q}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \longrightarrow 0,$$

which implies that $E_2^{0,q}(\mathbb{F}_p) \cong E_2^{0,q}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$. By tensoring (5.7) by $\mathbb{Q}$, we also have $E_2^{0,q}(\mathbb{Q}) \cong E_2^{0,q}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. By the fundamental theorem of finitely generated abelian groups, $\dim_{\mathbb{Q}} E_2^{0,q}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} < \dim_{\mathbb{F}_p} E_2^{0,q}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$, which contradicts to the fact that both $E_2^{0,q}(\mathbb{Q})$ and $E_2^{0,q}(\mathbb{F}_p)$ have the same rank (5.6). Hence, $E_2^{0,q}(\mathbb{Z})$ has no torsion element. By using the fundamental theorem of finitely generated abelian groups again, we see that $E_2^{0,q}(\mathbb{Z})$ is a free module of rank (5.6) over $\mathbb{Z}$.

Let us consider the case that R is an arbitrary commutative ring. By tensoring (5.7) by R, we have $E_2^{0,q}(R) \cong E_2^{0,q}(\mathbb{Z}) \otimes_{\mathbb{Z}} R$. Since $E_2^{0,q}(\mathbb{Z})$ is a free module of rank (5.6) over $\mathbb{Z}$, $E_2^{0,q}(R)$ is also a free module of rank (5.6) over R. This completes the proof. □

Theorem 5.11. *Let $m \geq 3$. The cohomology group $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ is a free module over R for $n \geq 0$. The rank of $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ is given by*

$$\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) = (m-1)\varphi(n) + \sum_{k=1}^{m-1} (-1)^{m+k} k\varphi(n-m+k).$$

Proof. The spectral sequence collapses from the E_2-page and there is no extension problem by Corollary 5.4. By Lemma 5.10, $E_2^{0,q}$ is a free module over R. The statement follows from that

$$\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) \cong E_2^{0,n}$$

and (5.6). □

Recall

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} (\mathrm{rank}_R \mathrm{N}_m(R)_n)\, t^n = 1 + \sum_{k=1}^{m-1} k t^{m-k}, \\ f^!(t) &= \sum_{n=0}^{\infty} \left(\mathrm{rank}_R \mathrm{N}_m(R)^!_n\right) t^n = \frac{1}{f(-t)} = \frac{1}{1 + \displaystyle\sum_{k=1}^{m-1} (-1)^{m-k} k t^{m-k}} \end{aligned}$$

in (4.5), (4.6), (4.4), and Proposition 4.15. Let us define

$$h(t) = \sum_{n=0}^{\infty} (\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)))\, t^n.$$

Theorem 5.12. *Let $m \geq 3$. The generating function $h(t)$ is given by*

$$h(t) = 1 + (m-2) f^!(t).$$

Proof. By Theorem 5.11,

$$\begin{aligned} h(t) &= \left\{(m-1)+\sum_{k=1}^{m-1}(-1)^{m+k}kt^{m-k}\right\}f^!(t) \\ &= \frac{(m-1)+\sum_{k=1}^{m-1}(-1)^{m-k}kt^{m-k}}{1+\sum_{k=1}^{m-1}(-1)^{m-k}kt^{m-k}} \\ &= 1+\frac{m-2}{1+\sum_{k=1}^{m-1}(-1)^{m-k}kt^{m-k}} \\ &= 1+(m-2)f^!(t). \end{aligned}$$

This completes the proof. □

Corollary 5.13. *Let $m \geq 3$. The rank of* $\mathrm{rank}_R\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ *is given by*

$$\mathrm{rank}_R\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) = \begin{cases} m-1 & (n=0), \\ (m-2)\varphi(n) & (n>0). \end{cases}$$

Proof. Note that the constant term of $f^!(t)$ is 1. The statement follows from Theorem 5.12. □

Remark 5.14. If $m=3$, then $\varphi(n)=n+1$ for $n \geq 0$. We can easily check that the result of Corollary 5.13 is compatible with [9, Theorem 5.4]:

$$\mathrm{rank}_R\mathrm{HH}^n(\mathrm{N}_3(R), \mathrm{M}_3(R)/\mathrm{N}_3(R)) = \begin{cases} 2 & (n=0), \\ n+1 & (n>0). \end{cases}$$

Recall that $\mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) = H^n(C^{*,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)))$ (cf. (5.1)). By the result above, we have:

Theorem 5.15. *Let $m \geq 3$. For each $n \geq 0$ and $s \in \mathbb{Z}$, $\mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ is a free R-module. The rank is given by*

$$\mathrm{rank}_R\mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) = \begin{cases} 0 & (n \neq s), \\ m-1 & (n=s=0), \\ (m-2)\varphi(n) & (n=s>0). \end{cases}$$

Proof. As in Corollary 5.4, we have

$$\begin{aligned} \mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) &\cong E_\infty^{n-s,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) \\ &\cong E_2^{n-s,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)). \end{aligned}$$

By the discussion above, we can verify the statement. □

As in the proofs of Claim 5.9, Lemma 5.10, and Theorem 5.11, we can show the following:

Proposition 5.16. *Let $m \geq 3$. The cohomology group $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/J(\mathrm{N}_m(R)))$ is a free module over R for $n \geq 0$. The rank of $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/J(\mathrm{N}_m(R)))$ is given by*

$$\mathrm{rank}_R\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/J(\mathrm{N}_m(R))) = \begin{cases} m & (n=0), \\ (m-1)\varphi(n) & (n>0). \end{cases}$$

For each $n \geq 0$ and $s \in \mathbb{Z}$, $\mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/J(\mathrm{N}_m(R)))$ is a free R-module. The rank is given by

$$\mathrm{rank}_R \mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{M}_m(R)/J(\mathrm{N}_m(R))) = \begin{cases} 0 & (n \neq s), \\ m & (n = s = 0), \\ (m-1)\varphi(n) & (n = s > 0). \end{cases}$$

5.3. **The Zariski tangent space of the moduli of subalgebras of M_m at N_m.** In the previous subsection, we have calculated $\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$. In this subsection, we calculate the dimension of the tangent space of the moduli of subalgebras of M_m over $\mathbb{Z}$ at N_m for $m \geq 3$ by using $\mathrm{rank}_R \mathrm{HH}^1(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$.

Proposition 5.17. *Let R be a commutative ring. Set*

$$N(\mathrm{N}_m(R)) = \{A \in \mathrm{M}_m(R) \mid [A, B] := AB - BA \in \mathrm{N}_m(R) \text{ for any } B \in \mathrm{N}_m(R)\}.$$

For $m \geq 3$, $N(\mathrm{N}_m(R)) = \mathrm{B}_m(R)$.

Proof. It is easy to see that $N(\mathrm{N}_m(R)) \supseteq \mathrm{B}_m(R)$. Let us show that $N(\mathrm{N}_m(R)) \subseteq \mathrm{B}_m(R)$. Recall the $\mathbb{Z}$-grading on $\mathrm{M} = \mathrm{M}_m(R)$ in §5.1:

$$\mathrm{M} = \bigoplus_{r \in \mathbb{Z}} \mathrm{M}_r, \text{ where } \mathrm{M}_r = \bigoplus_{j-i=r} R\{E_{i,j}\}.$$

Set $N(\mathrm{N}_m(R))_r = N(\mathrm{N}_m(R)) \cap \mathrm{M}_r$. Since $N(\mathrm{N}_m(R)) = \oplus_{r \in \mathbb{Z}} N(\mathrm{N}_m(R))_r$ and $\mathrm{B}_m(R) = \oplus_{r \geq 0} \mathrm{M}_r$, it suffices to prove that $N(\mathrm{N}_m(R))_r = 0$ for $-(m-1) \leq r \leq -1$. Suppose that there exists $x = a_1 E_{1-r,1} + a_2 E_{2-r,2} + \cdots + a_{m+r} E_{m,m+r} \in N(\mathrm{N}_m(R))_r$ with $a_1 = \cdots = a_{i-1} = 0$ and $a_i \neq 0$. If $-(m-1) \leq r \leq -2$, then

$$[x, E_{i,i+1}] = a_i E_{i-r,i+1} \in \mathrm{N}_m(R).$$

This implies that $a_i = 0$, which is a contradiction. If $r = -1$, then

$$[x, E_{i,i+1}] = a_i E_{i+1,i+1} - a_i E_{i,i} \in \mathrm{N}_m(R).$$

Since $m \geq 3$, we see that $a_i = 0$, which is a contradiction. Hence, $N(\mathrm{N}_m(R)) = \mathrm{B}_m(R)$. □

Set $d = \mathrm{rank}_R \mathrm{N}_m(R) = \dfrac{m^2 - m + 2}{2}$. Recall the moduli of molds $\mathrm{Mold}_{m,d}$, in other words, the moduli of rank d subalgebras of the full matrix ring M_m in [9, §3.1]. We can regard N_m as a point of $\mathrm{Mold}_{m,d}$. Let us consider the Zariski tangent space $T_{\mathrm{Mold}_{m,d}/\mathbb{Z}, \mathrm{N}_m}$ of $\mathrm{Mold}_{m,d}$ over $\mathbb{Z}$ at N_m (for details, see [9, Definition 3.10]).

Theorem 5.18. *The dimension of the Zariski tangent space $T_{\mathrm{Mold}_{m,d}/\mathbb{Z}, \mathrm{N}_m}$ of $\mathrm{Mold}_{m,d}$ over $\mathbb{Z}$ at N_m is*

$$\dim T_{\mathrm{Mold}_{m,d}/\mathbb{Z}, \mathrm{N}_m} = \frac{3m^2 - 7m + 4}{2}$$

for $m \geq 3$.

Proof. Let $m \geq 3$. For any field k,

$$\dim_k \mathrm{HH}^1(\mathrm{N}_m(k), \mathrm{M}_m(k)/\mathrm{N}_m(k)) = (m-2)\varphi(1) = (m-2)(m-1)$$

by Corollaries 4.14 and 5.13. We also see that

$$\dim_k N(\mathrm{N}_m(k)) = \dim_k \mathrm{B}_m(k) = \frac{m(m+1)}{2}$$

by Proposition 5.17. Using [9, Corollary 3.14], we obtain

$$\begin{aligned}\dim T_{\mathrm{Mold}_{m,d}/\mathbb{Z},\mathrm{N}_m} &= \dim_k \mathrm{HH}^1(\mathrm{N}_m(k(x)), \mathrm{M}_m(k(x))/\mathrm{N}_m(k(x))) + m^2 - \dim_k N(\mathrm{N}_m(k(x))) \\ &= \frac{3m^2-7m+4}{2},\end{aligned}$$

where $k(x)$ is the residue field of $x = \mathrm{N}_m$. □

Remark 5.19. By Theorem 5.18, $\dim T_{\mathrm{Mold}_{3,4}/\mathbb{Z},\mathrm{N}_3} = 5$ for $m = 3$. This result coincides with $\dim T_{\mathrm{Mold}_{3,4}/\mathbb{Z},\mathrm{N}_3}$ in [9, Table 2].

Remark 5.20. In the case $m = 2$, $\mathrm{N}_2(R)$ coincides with $\mathrm{J}_2(R)$ defined in [9, Definition 4.16]. For any field k, we have obtained

$$\dim_k \mathrm{HH}^1(\mathrm{N}_2(k), \mathrm{M}_2(k)/\mathrm{N}_2(k)) = \begin{cases} 1 & (\mathrm{ch}(k) \neq 2), \\ 2 & (\mathrm{ch}(k) = 2) \end{cases}$$

by [9, Corollary 4.20]. We also see that

$$N(\mathrm{N}_2(k)) = \begin{cases} \mathrm{B}_2(k) & (\mathrm{ch}(k) \neq 2), \\ \mathrm{M}_2(k) & (\mathrm{ch}(k) = 2) \end{cases}$$

by [9, Proposition 4.21]. Using [9, Corollary 3.14] or [9, Example 4.22], we obtain

$$\dim T_{\mathrm{Mold}_{2,3}/\mathbb{Z},\mathrm{N}_2} = 2,$$

while $\mathrm{Mold}_{2,3} = \mathbb{P}^2_{\mathbb{Z}}$ ([9, Example 3.6]).

6. The R-module structure of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$

In this section, we determine the R-module structure of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ for $m \geq 3$. (The case $m = 2$ will be discussed in §9.) Throughout this section, we assume that $m \geq 3$. Set $\mathrm{N} = \mathrm{N}_m(R)$, $\mathrm{B} = \mathrm{B}_m(R)$, and $\mathrm{J} = J(\mathrm{N}_m(R))$. In §6.1, we consider a spectral sequence converging to the Hochschild cohomology $\mathrm{HH}^*(\mathrm{N}, \mathrm{N})$. In §6.2, we show that $E_2^{p,q}(\mathrm{B}) = 0$ unless $p = 0, m-1$, where $E_1^{p,q}(\mathrm{B}) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{B})$ is a spectral sequence converging to $\mathrm{HH}^*(\mathrm{N}, \mathrm{B})$. We also show that $E_2^{p,q}(\mathrm{B})$ is a finitely generated free module over R for $p = 0, m-1$. In §6.3, we calculate the rank of $E_2^{p,q}(\mathrm{B})$ over R for $p = 0, m-1$. In §6.4, we show that $E_2^{1,q}(\mathrm{N})$ is a finitely generated free module over R for the spectral sequence $E_1^{p,q}(\mathrm{N}, \mathrm{N}) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{N})$. In §6.5, we calculate the rank of $E_2^{p,q}(\mathrm{N})$ over R for any p. As a result, we determine the R-module structure of $\mathrm{HH}^*(\mathrm{N}, \mathrm{N})$.

6.1. **Degeneration of spectral sequences.** Recall the filtration $\{\overline{\mathrm{J}}^p\mathrm{N}\}$ in (3.1) or §5.1. Obviously, $\overline{\mathrm{J}}^p\mathrm{N} = \mathrm{J}^p$ as ideals of N. By regarding N as a subobject of the $\mathbb{Z}$-graded R-algebra $\mathrm{M} = \mathrm{M}_m(R)$, we have $F^p\mathrm{N} = \overline{\mathrm{J}}^p\mathrm{N}$, where $F^p\mathrm{N} = \mathrm{N} \cap F^p\mathrm{M}$ has been defined in §5.1. Using the filtration $\{\overline{\mathrm{J}}^p\mathrm{N}\}$, we have a spectral sequence

$${}^JE_1^{p,q}(\mathrm{N}, \mathrm{N}) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{N})$$

of R-algebras by Proposition 3.3. By the argument in §5, we see that it can be promoted to a spectral sequence of $\mathbb{Z}$-graded R-algebras.

Proposition 6.1. *The spectral sequence*

$${}^JE_1^{p,q}(\mathrm{N}, \mathrm{N}) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{N})$$

of $\mathbb{Z}$-graded R-algebras collapses from the E_2-page. There is an isomorphism

$$\mathrm{HH}^n(\mathrm{N}, \mathrm{N}) = \bigoplus_{n,s} \mathrm{HH}^{n,s}(\mathrm{N}, \mathrm{N}) \cong \bigoplus_{n,s} {}^JE_\infty^{n-s,s}(\mathrm{N}, \mathrm{N}) \tag{6.1}$$

of bigraded R-modules, where $\mathrm{HH}^{n,s}(\mathrm{N}, \mathrm{N}) \cong {}^JE_\infty^{n-s,s}(\mathrm{N}, \mathrm{N})$.

Proof. Since the filtration $\{\overline{\mathrm{J}}^p\mathrm{N}\}$ coincides with $\{F^p\mathrm{N}\}$, the spectral sequence $\{{}^JE_r^{p,q}(\mathrm{N},\mathrm{N})\}_{r\geq 1}$ is isomorphic to $\{{}^ME_r^{p,q}(\mathrm{N},\mathrm{N})\}_{r\geq 1}$ given by Proposition 5.1. Therefore, these spectral sequences collapse from the E_2-pages, and there are no extension problems as spectral sequences of $\mathbb{Z}$-graded R-modules by Theorem 5.3. This completes the proof. □

Remark 6.2. We can also prove that (6.1) is an isomorphism of bigraded R-algebras, which will be proved in §7.

In the sequel, we omit J of ${}^JE_1^{p,q}(\mathrm{N},\mathrm{N})$. We also write $E_r^{p,q}(\mathrm{N}) = E_r^{p,q}(\mathrm{N},\mathrm{N})$ and $E_r^{p,q,s}(\mathrm{N}) = E_r^{p,q,s}(\mathrm{N},\mathrm{N})$. Here we rephrase Lemma 5.2, which will be used later.

Lemma 6.3. *We have*

$$E_1^{p,q,s}(\mathrm{N}) \cong \begin{cases} \mathrm{N}^!_{p+q} \otimes_R \mathrm{Gr}^p(\mathrm{N}) & (q = s), \\ 0 & (q \neq s). \end{cases}$$

When we consider B as a $\mathbb{Z}$-graded N-bimodule, we can obtain a spectral sequence

$$E_1^{p,q}(\mathrm{B}) = \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{Gr}^p(\mathrm{B})) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{B})$$

by Proposition 5.1. For the $\mathbb{Z}$-graded N-bimodule B/N, we also obtain a spectral sequence

$$E_1^{p,q}(\mathrm{B}/\mathrm{N}) = \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{Gr}^p(\mathrm{B}/\mathrm{N})) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{B}/\mathrm{N}).$$

Then there exist morphisms of spectral sequences

$$E_r^{*,*}(\mathrm{N}) \longrightarrow E_r^{*,*}(\mathrm{B}) \longrightarrow E_r^{*,*}(\mathrm{B}/\mathrm{N}).$$

Note that $E_1^{p,q}(\mathrm{N}), E_1^{p,q}(\mathrm{B})$, and $E_1^{p,q}(\mathrm{B}/\mathrm{N})$ are finitely generated free modules over R. Indeed, for example, $E_1^{p,q}(\mathrm{B}) \cong \mathrm{N}^!_{p+q} \otimes_R \mathrm{Gr}^p(\mathrm{B})$ is a finitely generated free module over R. Unless $0 \leq p \leq m-1$, $E_r^{p,q}(\mathrm{N}) = E_r^{p,q}(\mathrm{B}) = E_r^{p,q}(\mathrm{B}/\mathrm{N}) = 0$ for $r \geq 1$, since $\mathrm{Gr}^p(\mathrm{N}) = \mathrm{Gr}^p(\mathrm{B}) = \mathrm{Gr}^p(\mathrm{B}/\mathrm{N}) = 0$.

Let us describe results on $E_r^{p,q}(\mathrm{B}/\mathrm{N})$ and $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{B}_m(R)/\mathrm{N}_m(R))$. Using

$$\mathrm{Gr}^p(\mathrm{B}/\mathrm{N}) = \begin{cases} \mathrm{B}/\mathrm{N} & (p = 0), \\ 0 & (p \neq 0), \end{cases}$$

we obtain the following theorem.

Theorem 6.4. *For $r \geq 1$,*

$$E_r^{p,q}(\mathrm{B}/\mathrm{N}) = \begin{cases} \mathrm{N}^!_q \otimes_R (\mathrm{B}/\mathrm{N}) & (p = 0), \\ 0 & (p \neq 0). \end{cases}$$

For $n \geq 0$,

$$\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{B}_m(R)/\mathrm{N}_m(R)) = \mathrm{N}^!_n \otimes_R (\mathrm{B}/\mathrm{N}) \cong \mathrm{N}^!_n \otimes_R R^{m-1}.$$

In the sequel, if we emphasize the commutative ring R, then we write $E_r^{p,q}(\mathrm{N}; R) = E_r^{p,q}(\mathrm{N})$, $E_r^{p,q}(\mathrm{B}; R) = E_r^{p,q}(\mathrm{B})$, and $E_r^{p,q}(\mathrm{B}/\mathrm{N}; R) = E_r^{p,q}(\mathrm{B}/\mathrm{N})$, respectively.

6.2. **The freeness of $E_2^{p,q}(\mathrm{B})$.** In this subsection, we show that $E_2^{p,q}(\mathrm{B}) = 0$ unless $p = 0, m-1$. We also show that $E_2^{p,q}(B)$ is a finitely generated free module over R for $p = 0, m-1$. Note that $E_1^{p,q}(\mathrm{B}) = \mathrm{N}^!_{p+q} \otimes_R \mathrm{Gr}^p(\mathrm{B})$ is a finitely generated free module over R for any p.

For $0 \le p \le m-2$,

$$\begin{array}{ccl} E_1^{p,q}(\mathrm{B}) = \mathrm{N}^!_{p+q} \otimes \mathrm{Gr}^p(\mathrm{B}) & \xrightarrow{d_1^{p,q}} & E_1^{p+1,q}(\mathrm{B}) = \mathrm{N}^!_{p+q+1} \otimes \mathrm{Gr}^{p+1}(\mathrm{B}) \\ f \otimes E_{i,i+p} & \longmapsto & \begin{cases} (-1)^{p+q+1} f y_{p+1} \otimes E_{1,p+2} & (i=1), \\ y_{i-1} f \otimes E_{i-1,i+p} \\ \quad +(-1)^{p+q+1} f y_{i+p} \otimes E_{i,i+p+1} & (2 \le i \le m-p-1), \\ y_{m-p-1} f \otimes E_{m-p-1,m} & (i = m-p) \end{cases} \end{array}$$

for $1 \le i \le m-p$.

Definition 6.5. For $1 \le p \le m-1$, we define an R-homomorphism $s^{p,q} : E_1^{p,q} \to E_1^{p-1,q}$ by

$$\begin{array}{ccl} E_1^{p,q}(\mathrm{B}) = \mathrm{N}^!_{p+q} \otimes \mathrm{Gr}^p(\mathrm{B}) & \xrightarrow{s^{p,q}} & E_1^{p-1,q}(\mathrm{B}) = \mathrm{N}^!_{p+q-1} \otimes \mathrm{Gr}^{p-1}(\mathrm{B}) \\ f \otimes E_{i,i+p} & \longmapsto & \begin{cases} f_R \otimes E_{i+1,i+p} & (f = y_i f_R, f_R \in \mathrm{N}^!_{p+q-1}, \\ & \quad 1 \le i \le m-p), \\ (-1)^{p+q} f_L \otimes E_{1,p} & (i = 1, f \notin y_1 \mathrm{N}^!_{p+q-1}, \\ & \quad f = f_L y_p, f_L \in \mathrm{N}^!_{p+q-1}), \\ 0 & (\text{otherwise}) \end{cases} \end{array}$$

for $1 \le i \le m-p$ and $f \in \mathcal{B}(\mathrm{N}^!_{p+q})$.

Lemma 6.6. *For $1 \le p \le m-2$, $s^{p+1,q} \circ d_1^{p,q} + d_1^{p-1,q} \circ s^{p,q} = id_{E_1^{p,q}(\mathrm{B})}$.*

Proof. Let $1 \le p \le m-2$. For $1 \le i \le m-p$ and $f \in \mathcal{B}(\mathrm{N}^!_{p+q})$,

$$\begin{aligned} & s^{p+1,q} \circ d_1^{p,q}(f \otimes E_{i,i+p}) \\ = & \begin{cases} (-1)^{p+q+1} f_R y_{p+1} \otimes E_{2,p+2} & (i = 1, f = y_1 f_R, f_R \in \mathrm{N}^!_{p+q-1}), \\ 0 & (i=1, f \notin y_1 \mathrm{N}^!_{p+q-1}, f = f_L y_p, f_L \in \mathrm{N}^!_{p+q-1}), \\ f \otimes E_{1,p+1} & (i = 1, f \notin y_1 \mathrm{N}^!_{p+q-1}, f \notin \mathrm{N}^!_{p+q-1} y_p), \\ (-1)^{p+q+1} f_R y_{i+p} \otimes E_{i+1,i+p+1} & (2 \le i \le m-p-1, f = y_i f_R, f_R \in \mathrm{N}^!_{p+q-1}), \\ f \otimes E_{i,i+p} & (2 \le i \le m-p-1, f \notin y_i \mathrm{N}^!_{p+q-1}), \\ 0 & (i = m-p, f = y_{m-p} f_R, f_R \in \mathrm{N}^!_{p+q-1}), \\ f \otimes E_{m-p,m} & (i = m-p, f \notin y_{m-p} \mathrm{N}^!_{p+q-1}). \end{cases} \end{aligned}$$

For $1 \le i \le m-p$ and $f \in \mathcal{B}(\mathrm{N}^!_{p+q})$,

$$\begin{aligned} & d_1^{p-1,q} \circ s^{p,q}(f \otimes E_{i,i+p}) \\ = & \begin{cases} f \otimes E_{1,p+1} + (-1)^{p+q} f_R y_{p+1} \otimes E_{2,p+2} & (i = 1, f = y_1 f_R, f_R \in \mathrm{N}^!_{p+q-1}), \\ f \otimes E_{1,p+1} & (i=1, f \notin y_1 \mathrm{N}^!_{p+q-1}, f = f_L y_p, f_L \in \mathrm{N}^!_{p+q-1}), \\ 0 & (i = 1, f \notin y_1 \mathrm{N}^!_{p+q-1}, f \notin \mathrm{N}^!_{p+q-1} y_p), \\ f \otimes E_{i,i+p} + (-1)^{p+q} f_R y_{i+p} \otimes E_{i+1,i+p+1} & (2 \le i \le m-p-1, f = y_i f_R, f_R \in \mathrm{N}^!_{p+q-1}), \\ 0 & (2 \le i \le m-p-1, f \notin y_i \mathrm{N}^!_{p+q-1}), \\ f \otimes E_{m-p,m} & (i = m-p, f = y_{m-p} f_R, f_R \in \mathrm{N}^!_{p+q-1}), \\ 0 & (i = m-p, f \notin y_{m-p} \mathrm{N}^!_{p+q-1}). \end{cases} \end{aligned}$$

Hence, we see that $s^{p+1,q} \circ d_1^{p,q} + d_1^{p-1,q} \circ s^{p,q} = id_{E_1^{p,q}(\mathrm{B})}$. □

Proposition 6.7. *For $1 \le p \le m-2$, $E_2^{p,q}(\mathrm{B}; R) = 0$.*

Proof. The statement follows from Lemma 6.6. $\square$

By Proposition 6.7, $E_2^{p,q}(\mathrm{B};R) = 0$ unless $p = 0, m-1$. For $p = 0, m-1$, we have the following proposition:

Proposition 6.8. *For $p = 0, m-1$, $E_2^{p,q}(\mathrm{B};R)$ is a finitely generated free R-module.*

Proof. Let $p = 0$. For $R = \mathbb{Z}$, $E_2^{0,q}(\mathrm{B};\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module, since $E_2^{0,q}(\mathrm{B};\mathbb{Z}) \cong \mathrm{Ker} d_1^{0,q}$ is a submodule of the finitely generated free $\mathbb{Z}$-module $E_1^{0,q}(\mathrm{B};\mathbb{Z}) \cong \mathrm{N}^!_q \otimes_{\mathbb{Z}} \mathrm{Gr}^0(\mathrm{B})$. For an arbitrary commutative ring R, we have an exact sequence

$$0 \longrightarrow E_2^{0,q}(\mathrm{B};\mathbb{Z}) \otimes_{\mathbb{Z}} R \longrightarrow E_2^{0,q}(\mathrm{B};R) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(E_2^{1,q}(\mathrm{B};\mathbb{Z}), R)$$

by the universal coefficient theorem. By Proposition 6.7, $E_2^{1,q}(\mathrm{B};\mathbb{Z}) = 0$ for $m \geq 3$. Hence, $E_2^{0,q}(\mathrm{B};\mathbb{Z}) \otimes_{\mathbb{Z}} R \cong E_2^{0,q}(\mathrm{B};R)$. Therefore, $E_2^{0,q}(\mathrm{B};R)$ is a finitely generated free R-module.

Let $p = m-1$. By Lemma 6.6 and the diagram

$$\begin{array}{ccccccc}
E_1^{m-3,q}(\mathrm{B};R) & \xrightarrow{d_1^{m-3,q}} & E_1^{m-2,q}(\mathrm{B};R) & \xrightarrow{d_1^{m-2,q}} & E_1^{m-1,q}(\mathrm{B};R) & \longrightarrow & 0 \\
\downarrow{\scriptstyle id} & \swarrow{\scriptstyle s^{m-2,q}} & \downarrow{\scriptstyle id} & \swarrow{\scriptstyle s^{m-1,q}} & \downarrow{\scriptstyle id} & & \\
E_1^{m-3,q}(\mathrm{B};R) & \xrightarrow{d_1^{m-3,q}} & E_1^{m-2,q}(\mathrm{B};R) & \xrightarrow{d_1^{m-2,q}} & E_1^{m-1,q}(\mathrm{B};R) & \longrightarrow & 0,
\end{array}$$

we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Im}\, d_1^{m-2,q} \longrightarrow E_1^{m-1,q}(\mathrm{B};R) \longrightarrow E_2^{m-1,q}(\mathrm{B};R) \longrightarrow 0,$$

which is split by $s^{m-1,q}$. Hence, $E_1^{m-1,q}(\mathrm{B};R) \cong \mathrm{Im}\, d_1^{m-2,q} \oplus E_2^{m-1,q}(\mathrm{B};R)$. For $R = \mathbb{Z}$, $E_2^{m-1,q}(\mathrm{B};\mathbb{Z})$ is finitely generated free $\mathbb{Z}$-module, since it is isomorphic to a submodule of the finitely generated free $\mathbb{Z}$-module $E_1^{m-1,q}(\mathrm{B};\mathbb{Z}) \cong \mathrm{N}^!_{m+q-1} \otimes_{\mathbb{Z}} \mathrm{Gr}^{m-1}(\mathrm{B})$. For an arbitrary commutative ring R, we have an exact sequence

$$0 \longrightarrow E_2^{m-1,q}(\mathrm{B};\mathbb{Z}) \otimes_{\mathbb{Z}} R \longrightarrow E_2^{m-1,q}(\mathrm{B};R) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(E_2^{m,q}(\mathrm{B};\mathbb{Z}), R)$$

by the universal coefficient theorem. By $E_2^{m,q}(\mathrm{B};\mathbb{Z}) = 0$, $E_2^{m-1,q}(\mathrm{B};R) \cong E_2^{m-1,q}(\mathrm{B};\mathbb{Z}) \otimes_{\mathbb{Z}} R$. Hence, $E_2^{m-1,q}(\mathrm{B};R)$ is a finitely generated free R-module. $\square$

Corollary 6.9. *For $m \geq 4$ and $2 \leq p \leq m-2$, $E_2^{p,q}(\mathrm{N};R) = 0$. For $m \geq 3$, $E_2^{m-1,q}(\mathrm{N};R)$ is isomorphic to the finitely generated free R-module $E_2^{m-1,q}(\mathrm{B};R)$.*

Proof. Since $E_1^{1,q}(\mathrm{N}) \to E_1^{2,q}(\mathrm{N}) \to \cdots \to E_1^{m-1,q}(\mathrm{N}) \to 0$ is isomorphic to $E_1^{1,q}(\mathrm{B}) \to E_1^{2,q}(\mathrm{B}) \to \cdots \to E_1^{m-1,q}(\mathrm{B}) \to 0$, $E_2^{p,q}(\mathrm{N};R) \cong E_2^{p,q}(\mathrm{B};R)$ for $2 \leq p \leq m-1$. The statements follow from Propositions 6.7 and 6.8. $\square$

Remark 6.10. Recall that $E_2^{p,q}(\mathrm{N}) = 0$ unless $0 \leq p \leq m-1$. Furthermore, we also see that $E_2^{p,q}(\mathrm{N}) = 0$ unless $p = 0, 1, m-1$ for $m \geq 3$ by Corollary 6.9.

Definition 6.11. Let $q > 0$. For $I = (i_1, i_2, \ldots, i_q)$ with $1 \leq i_1, i_2, \ldots, i_q \leq m-1$, set $y_I = y_{i_1} y_{i_2} \cdots y_{i_q} \in \mathrm{N}^!_q$. For $I = (i_1, i_2, \ldots, i_q)$, we define the length $|I|$ of I by q. We also define

$$z(i, I) = y_i y_I \otimes E_{i,i} + (-1)^{q+1} y_I y_i \otimes E_{i+1,i+1} \in \mathrm{N}^!_{q+1} \otimes_R \mathrm{Gr}^0(\mathrm{B}) \cong E_1^{0,q+1}(B)$$

for $1 \leq i \leq m-1$ and I with $|I| = q$.

Let $q = 0$. We define

$$z(i, \emptyset) = z(i) = y_i \otimes E_{i,i} - y_i \otimes E_{i+1,i+1} \in \mathrm{N}^!_1 \otimes_R \mathrm{Gr}^0(\mathrm{B}) \cong E_1^{0,1}(B)$$

for $1 \le i \le m-1$. If $I = \emptyset$, then put $y_\emptyset = 1$ and $|\emptyset| = 0$. For $q \ge 0$, set

$$\mathcal{Z}(q) = \{z(i, I) \mid i = 1, 2, \dots, m-1, |I| = q\}.$$

Proposition 6.12. *As an R-submodule of $E_1^{0,0}(\mathrm{B}; R)$, $E_2^{0,0}(\mathrm{B}; R) = RI_m$. For $q \ge 0$, $E_2^{0,q+1}(\mathrm{B}; R) = R\{z \mid z \in \mathcal{Z}(q)\}$ as R-submodules of $E_1^{0,q+1}(\mathrm{B}; R) \cong \mathrm{N}^!_{q+1} \otimes_R \mathrm{Gr}^0(\mathrm{B})$.*

Proof. Note that

$$\begin{array}{ccc} E_1^{0,q+1}(\mathrm{B}; R) \cong \mathrm{N}^!_{q+1} \otimes_R \mathrm{Gr}^0(\mathrm{B}) & \stackrel{d_1^{0,q+1}}{\longrightarrow} & E_1^{1,q+1}(\mathrm{B}; R) \cong \mathrm{N}^!_{q+2} \otimes_R \mathrm{Gr}^1(\mathrm{B}) \\ f \otimes E_{i,i} & \longmapsto & \begin{cases} (-1)^q f y_1 \otimes E_{1,2} & (i = 1), \\ \begin{array}{l} y_{i-1} f \otimes E_{i-1,i} \\ +(-1)^q f y_i \otimes E_{i,i+1} \end{array} & (2 \le i \le m-1), \\ y_{m-1} f \otimes E_{m-1,m} & (i = m) \end{cases} \end{array}$$

for $f \in \mathrm{N}^!_{q+1}$. Let $z = \sum_{i=1}^m f_i \otimes E_{i,i} \in E_1^{0,q+1}(\mathrm{B}; R)$ with $f_i \in \mathrm{N}^!_{q+1}$ $(1 \le i \le m)$. Since

$$d_1^{0,q+1}(z) = \{(-1)^q f_1 y_1 + y_1 f_2\} \otimes E_{1,2} + \cdots + \{(-1)^q f_{m-1} y_{m-1} + y_{m-1} f_m\} \otimes E_{m-1,m},$$

$z \in \mathrm{Ker} d_1^{0,q+1}$ if and only if

$$(6.2) \qquad (-1)^{q+1} f_1 y_1 = y_1 f_2, (-1)^{q+1} f_2 y_2 = y_2 f_3, \dots, (-1)^{q+1} f_{m-1} y_{m-1} = y_{m-1} f_m.$$

When $q = -1$, $z \in \mathrm{Ker} d_1^{0,0}$ if and only if $f_1 = f_2 = \cdots = f_m \in R$. Hence, $E_2^{0,0}(\mathrm{B}; R) \cong \mathrm{Ker} d_1^{0,0} = RI_m$.

Let us consider $E_2^{0,q+1} = \mathrm{Ker} d_1^{0,q+1} \subseteq E_1^{0,q+1} \cong \mathrm{N}^!_{q+1} \otimes_R \mathrm{Gr}^0(\mathrm{B})$ for $q \ge 0$. It is easy to verify that $z(i, I) \in \mathrm{Ker} d_1^{0,q+1}$. Conversely, suppose that $z = \sum_{i=1}^m f_i \otimes E_{i,i} \in \mathrm{Ker} d_1^{0,q+1} \subseteq E_1^{0,q+1}(\mathrm{B}; R)$ with $f_i \in \mathrm{N}^!_{q+1}$ $(1 \le i \le m)$. By (6.2), there exists $g_1, g_2, \dots, g_{m-1} \in \mathrm{N}^!_q$ such that

$$\begin{array}{rcl} f_1 & = & y_1 g_1, \\ f_2 & = & (-1)^{q+1} g_1 y_1 + y_2 g_2, \\ \cdots & \cdots & \cdots \\ f_{m-1} & = & (-1)^{q+1} g_{m-2} y_{m-2} + y_{m-1} g_{m-1}, \\ f_m & = & (-1)^{q+1} g_{m-1} y_{m-1}. \end{array}$$

Then

$$z = \{y_1 g_1 \otimes E_{1,1} + (-1)^{q+1} g_1 y_1 \otimes E_{2,2}\} + \cdots + \{y_{m-1} g_{m-1} \otimes E_{m-1,m-1} + (-1)^{q+1} g_{m-1} y_{m-1} \otimes E_{m,m}\}.$$

Hence, $z \in R\{z \mid z \in \mathcal{Z}(q)\}$. This completes the proof. □

Corollary 6.13. *As R-modules, $E_2^{0,1}(\mathrm{B}; R) \cong R^{m-1}$.*

Proof. By Proposition 6.12,

$$E_2^{0,1}(\mathrm{B}; R) = R\{z(1), z(2), \dots, z(m-1)\}.$$

It is easy to see that $z(1), z(2), \dots, z(m-1)$ are linearly independent over R. Hence, $E_2^{0,1}(\mathrm{B}; R) \cong R^{m-1}$. □

Proposition 6.14. *As quotient modules of $E_1^{m-1,-(m-1)+q}(\mathrm{B}) \cong \mathrm{N}^!_q \otimes_R \mathrm{Gr}^{m-1}(\mathrm{B})$ over R,*

$$E_2^{m-1,-(m-1)+q}(\mathrm{B}) = R\{y_I \otimes E_{1,m} \mid I = (i_1, \dots, i_q), i_1 \ne 1, i_q \ne m-1\}$$

for $q \ge 0$. Here, we understand that $E_2^{m-1,-(m-1)}(\mathrm{B}) = R\{1 \otimes E_{1,m}\}$ when $q = 0$. Moreover, $\{y_I \otimes E_{1,m} \mid I = (i_1, \dots, i_q), i_1 \ne 1, i_q \ne m-1\}$ are linearly independent in $E_2^{m-1,-(m-1)+q}(\mathrm{B})$ over R.

Proof. Let us consider

$$d_1^{m-2,-(m-1)+q} : E_1^{m-2,-(m-1)+q} \cong \mathrm{N}_{q-1} \otimes_R \mathrm{Gr}^{m-2}(\mathrm{B}) \longrightarrow E_1^{m-1,-(m-1)+q} \cong \mathrm{N}_q \otimes_R \mathrm{Gr}^{m-1}(\mathrm{B}).$$

Since

$$\begin{aligned} d_1^{m-2,-(m-1)+q}(y_J \otimes E_{1,m-1}) &= (-1)^q y_J y_{m-1} \otimes E_{1,m}, \\ d_1^{m-2,-(m-1)+q}(y_J \otimes E_{2,m}) &= y_1 y_J \otimes E_{1,m} \end{aligned}$$

for $|J| = q-1$, we have

$$\mathrm{Im}\, d_1^{m-2,-(m-1)+q} = R\{y_I \otimes E_{1,m} |\ I = (i_1, \ldots, i_q), i_1 = 1 \text{ or } i_q = m-1\}.$$

Hence, we can see that the statement is true. □

6.3. **The rank of $E_2^{p,q}(\mathrm{B})$.** In this subsection, we calculate the rank of the free R-module $E_2^{p,q}(\mathrm{B})$. We note that $\mathrm{rank}_R E_2^{p,q}(\mathrm{B}) = 0$ unless $p = 0$ or $m-1$ by Proposition 6.7. Recall

$$\varphi(q) = \mathrm{rank}_R \mathrm{N}_q^!.$$

for $q \in \mathbb{Z}$. We shall show the following theorem:

Theorem 6.15. *For each $q \in \mathbb{Z}$, we have*

$$\mathrm{rank}_R E_2^{p,q}(\mathrm{B}) = \begin{cases} \varphi(q) & (p = 0), \\ (-1)^m \varphi(q) + \displaystyle\sum_{k=0}^{m-1} (-1)^k (k+1) \varphi(q+m-k-1) & (p = m-1). \end{cases}$$

6.3.1. *The rank of $E_2^{0,q}(\mathrm{B})$.* First, we calculate the rank of $E_2^{0,q}(\mathrm{B})$. In this subsubsection, we show the following theorem, which claims the formula of $\mathrm{rank}_R E_2^{0,q}$ in Theorem 6.15.

Theorem 6.16. *For each $q \in \mathbb{Z}$, we have*

$$\mathrm{rank}_R E_2^{0,q}(\mathrm{B}) = \varphi(q).$$

Recall y_I in Definition 6.11. For $q \geq 0$, we have $\varphi(q) = \sharp\mathcal{B}(q)$, where

$$\mathcal{B}(q) = \{0 \neq y_I \in \mathrm{N}_q^! |\ I = (i_1, \ldots, i_q)\}.$$

We put

$$\begin{aligned} \varphi(q; i_1 \neq a) &= \sharp\mathcal{B}(q, i_1 \neq a), \\ \varphi(q; i_1 \neq a, i_q = b) &= \sharp\mathcal{B}(q, i_1 \neq a, i_q = b), \end{aligned}$$

and so on, where

$$\begin{aligned} \mathcal{B}(q; i_1 \neq a) &= \{y_I \in \mathcal{B}(q) |\ i_1 \neq a\}, \\ \mathcal{B}(q; i_1 \neq a, i_q = b) &= \{y_I \in \mathcal{B}(q) |\ i_1 \neq a, i_q = b\}, \end{aligned}$$

and so on. For $\varphi(q; i_1 \neq 1)$, we have the following lemma.

Lemma 6.17. *For $q \geq 0$, we have*

$$\varphi(q; i_1 \neq 1) = \sum_{r=0}^{m-1} (-1)^r \varphi(q-r),$$

where $\varphi(0; i_1 \neq 1) = 1$.

Proof. When $q = 0$, we have

$$\varphi(0; i_1 \neq 1) = 1 = \varphi(0).$$

We consider the case when $0 < q \leq m-1$. We have

$$\begin{aligned}
& \varphi(q; i_1 \neq 1) \\
= \;& \varphi(q) - \varphi(q; i_1 = 1) \\
= \;& \varphi(q) - \varphi(q-1; i_1 \neq 2) \\
= \;& \varphi(q) - \varphi(q-1) + \varphi(q-1; i_1 = 2) \\
= \;& \varphi(q) - \varphi(q-1) + \varphi(q-2; i_1 \neq 3) \\
= \;& \cdots \\
= \;& \sum_{s=0}^{t-1} (-1)^s \varphi(q-s) + (-1)^t \varphi(q-t; i_1 \neq t+1) \\
= \;& \sum_{s=0}^{q-2} (-1)^s \varphi(q-s) + (-1)^{q-1} \varphi(1; i_1 \neq q) \\
= \;& \sum_{s=0}^{q} (-1)^s \varphi(q-s).
\end{aligned}$$

We consider the case when $q \geq m$. We have

$$\begin{aligned}
& \varphi(q; i_1 \neq 1) \\
= \;& \sum_{s=0}^{t-1} (-1)^s \varphi(q-s) + (-1)^t \varphi(q-t; i_1 \neq t+1) \\
= \;& \sum_{s=0}^{m-3} (-1)^s \varphi(q-s) + (-1)^{m-2} \varphi(q-(m-2); i_1 \neq m-1) \\
= \;& \sum_{s=0}^{m-1} (-1)^s \varphi(q-s).
\end{aligned}$$

This completes the proof. □

By Lemma 4.16, we have another formula for $\varphi(q; i_1 \neq 1)$.

Lemma 6.18. *For $q > 0$, we have*

$$\varphi(q; i_1 \neq 1) = \sum_{r=1}^{m-2} (-1)^{r-1} (m-1-r) \varphi(q-r).$$

Proof. By Lemma 6.17, we have

$$\varphi(q; i_1 \neq 1) = \varphi(q) + \sum_{r=1}^{m-1} (-1)^r \varphi(q-r).$$

Using Lemma 4.16, we obtain

$$\begin{aligned}
& \varphi(q; i_1 \neq 1) \\
= & \sum_{r=1}^{m-1}(-1)^{r-1}(m-r)\varphi(q-r) + \sum_{r=1}^{m-1}(-1)^{r}\varphi(q-r) \\
= & \sum_{r=1}^{m-1}(-1)^{r-1}(m-(r+1))\varphi(q-r) \\
= & \sum_{r=1}^{m-2}(-1)^{r-1}(m-1-r)\varphi(q-r).
\end{aligned}$$

This completes the proof. □

Now, we calculate $\mathrm{rank}_R E_2^{0,q}(\mathrm{B})$ for $q \geq 0$. Recall $z(i,I)$ in Definition 6.11. By Proposition 6.12,

$$E_2^{0,q+1}(\mathrm{B}) = R\{z|\ z \in \mathcal{Z}(q)\}$$

for $q \geq 0$, where

$$\mathcal{Z}(q) = \{z(i,I)|\ i = 1, \ldots, m-1, |I| = q\}.$$

For $1 \leq i \leq m-1$, we set

$$\begin{aligned}
\mathcal{Z}(q;i)_1 &= \{z(i,I) \in \mathcal{Z}(q)|\ i_1 \neq i+1, i_q \neq i-1\}, \\
\mathcal{Z}(q;i)_2 &= \{z(i,I) \in \mathcal{Z}(q)|\ i_1 \neq i+1, i_q = i-1\}, \\
\mathcal{Z}(q;i)_3 &= \{z(i,I) \in \mathcal{Z}(q)|\ i_1 = i+1, i_q \neq i-1\}, \\
\mathcal{Z}(q;i)_4 &= \{z(i,I) \in \mathcal{Z}(q)|\ i_1 = i+1, i_q = i-1\}.
\end{aligned}$$

Here we understand $\mathcal{Z}(q;1)_2 = \mathcal{Z}(q;m)_2 = \mathcal{Z}(q;0)_3 = \mathcal{Z}(q;m-1)_3 = \mathcal{Z}(q;1)_4 = \mathcal{Z}(q;m-1)_4 = \emptyset$. We have a decomposition

$$\mathcal{Z}(q) = \bigcup_{r=1}^{4}\bigcup_{i=1}^{m-1}\mathcal{Z}(q;i)_r.$$

For any $z(i,I) \in \mathcal{Z}(q;i)_4$, we have $z(i,I) = 0$. Note that

$$z(i,I) = y_i y_I \otimes E_{i,i}$$

for $z(i,I) \in \mathcal{Z}(q;i)_2$, and

$$z(i,I) = (-1)^{q+1} y_I y_i \otimes E_{i+1,i+1}$$

for $z(i,I) \in \mathcal{Z}(q;i)_3$. It is easy to see that

$$R\{z(i,I) \mid z(i,I) \in \mathcal{Z}(q;i)_2\} = R\{z(i-1,I) \mid z(i-1,I) \in \mathcal{Z}(q;i)_3\}$$

for $2 \leq i \leq m-1$ and $q \geq 0$. Hence, we have

$$E_2^{0,q+1}(\mathrm{B}) = R\{z|\ z \in \bigcup_{i=1}^{m-1}\mathcal{Z}(q;i)_1 \cup \bigcup_{i=2}^{m-1}\mathcal{Z}(q;i)_2\}.$$

Proposition 6.19. *For $q > 0$, we have*

$$\mathrm{rank}_R E_2^{0,q+1}(\mathrm{B}) = (m-1)\varphi(q) - \varphi(q; i_1 \neq 1).$$

Proof. We can easily see that

$$\{z \mid 0 \neq z \in \bigcup_{i=1}^{m-1} \mathcal{Z}(q;i)_1 \cup \bigcup_{i=2}^{m-1} \mathcal{Z}(q;i)_2\}$$

is linearly independent over R. Note that $\mathcal{Z}(q;1)_2 = \emptyset$. Hence, we have

$$\begin{aligned}
\mathrm{rank}_R E_2^{0,q+1}(\mathrm{B}) &= \sum_{r=1}^{2}\sum_{i=1}^{m-1} \sharp\{z \mid 0 \neq z \in \mathcal{Z}(q;i)_r\} \\
&= \sum_{r=1}^{3}\sum_{i=1}^{m-1} \sharp\{z \mid 0 \neq z \in \mathcal{Z}(q;i)_r\} - \sum_{i=1}^{m-1} \sharp\{z \mid 0 \neq z \in \mathcal{Z}(q;i)_3\} \\
&= \sum_{i=1}^{m-1} (\varphi(q) - \sharp\mathcal{B}(q;i_1 = i+1, i_q = i-1)) - \sum_{i=1}^{m-1} \sharp\mathcal{B}(q;i_1 = i+1, i_q \neq i-1) \\
&= (m-1)\varphi(q) - \sum_{i=1}^{m-1} \sharp\mathcal{B}(q;i_1 = i+1, i_q = i-1) \\
&\qquad - \sum_{i=1}^{m-1} \sharp\mathcal{B}(q;i_1 = i+1, i_q \neq i-1) \\
&= (m-1)\varphi(q) - \sum_{i=1}^{m-1} \sharp\mathcal{B}(q;i_1 = i+1) \\
&= (m-1)\varphi(q) - \varphi(q;i_1 \neq 1).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 6.16. For $q < 0$, we have

$$\mathrm{rank}_R E_2^{0,q}(\mathrm{B}) = 0 = \varphi(q).$$

For $q = 0$, we have

$$\mathrm{rank}_R E_2^{0,0}(\mathrm{B}) = 1 = \varphi(0)$$

by Proposition 6.12. For $q = 1$, we have

$$\mathrm{rank}_R E_2^{0,1}(\mathrm{B}) = m-1 = \varphi(1)$$

by Corollaries 4.14 and 6.13.

We assume that $q > 1$. By Proposition 6.19, Lemmas 4.16 and 6.18, we obtain

$$\begin{aligned}
\mathrm{rank}_R E_2^{0,q}(\mathrm{B}) &= (m-1)\varphi(q-1) - \varphi(q-1;i_1 \neq 1) \\
&= (m-1)\varphi(q-1) + \sum_{r=1}^{m-2} (-1)^r (m-1-r)\varphi(q-1-r) \\
&= \varphi(q).
\end{aligned}$$

Therefore, we have proved Theorem 6.16. □

6.3.2. *The rank of* $E_2^{m-1,q}(\mathrm{B})$. Next, we calculate the rank of the $E_2^{m-1,q}(\mathrm{B})$. In this subsubsection, we show the following theorem, which claims the equivalent formula of $\mathrm{rank}_R E_2^{m-1,q}$ in Theorem 6.15.

Theorem 6.20. *For each $q \in \mathbb{Z}$, we have*

$$\mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) = \sum_{k=0}^{m-1}(-1)^k(k+1)\varphi(q-k) + (-1)^m\varphi(q-m+1).$$

For $q < 0$, Theorem 6.20 is true since the both sides of the formula are 0. It suffices to prove Theorem 6.20 for $q \geq 0$. Let us consider the case when $0 \leq q < m-1$.

Proposition 6.21. *For $0 \leq q < m-1$, we have*

$$\mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) = \sum_{r=0}^{q}(-1)^r\varphi(q-r; i_1 \neq 1),$$

where $\varphi(0; i_1 \neq 1) = 1$.

Proof. By Proposition 6.14,

$$E_2^{m-1,-(m-1)+q}(\mathrm{B}) = R\{y_I \otimes E_{1,m} |\ I = (i_1, \dots, i_q), i_1 \neq 1, i_q \neq m-1\}.$$

Since $\{y_I \otimes E_{1,m} |\ I = (i_1, \dots, i_q), i_1 \neq 1, i_q \neq m-1, y_I \neq 0\}$ is linearly independent over R, we have

$$\begin{aligned}
& \mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) \\
= & \ \varphi(q; i_1 \neq 1, i_q \neq m-1) \\
= & \ \varphi(q; i_1 \neq 1) - \varphi(q; i_1 \neq 1, i_q = m-1) \\
= & \ \varphi(q; i_1 \neq 1) - \varphi(q-1; i_1 \neq 1; i_{q-1} \neq m-2) \\
= & \ \varphi(q; i_1 \neq 1) - \varphi(q-1; i_1 \neq 1) + \varphi(q-1; i_1 \neq 1, i_{q-1} = m-2) \\
= & \ \varphi(q; i_1 \neq 1) - \varphi(q-1; i_1 \neq 1) + \varphi(q-2; i_1 \neq 1, i_{q-2} \neq m-3) \\
= & \ \cdots \\
= & \ \sum_{r=0}^{s-1}(-1)^r\varphi(q-r; i_1 \neq 1) + (-1)^s\varphi(q-s; i_1 \neq 1, i_{q-s} \neq m-(s+1)) \\
= & \ \sum_{r=0}^{q-2}(-1)^r\varphi(q-r; i_1 \neq 1) + (-1)^{q-1}\varphi(1; i_1 \neq 1, i_1 \neq m-q) \\
= & \ \sum_{r=0}^{q}(-1)^r\varphi(q-r; i_1 \neq 1),
\end{aligned}$$

which is what we wanted. □

Proposition 6.22. *For $0 \leq q < m-1$, we have*

$$\mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) = \sum_{k=0}^{q}(-1)^k(k+1)\varphi(q-k).$$

Proof. By Lemma 6.17 and Proposition 6.21, we have

$$\begin{aligned}
\mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) &= \sum_{r=0}^{q}(-1)^r\sum_{s=0}^{q-r}(-1)^s\varphi(q-r-s) \\
&= \sum_{k=0}^{q}(-1)^k(k+1)\varphi(q-k).
\end{aligned}$$

This completes the proof. $\square$

Let us consider the case when $q = m-1$.

Proposition 6.23. *We have*

$$\mathrm{rank}_R E_2^{m-1,0}(\mathrm{B}) = \sum_{r=0}^{m-2} (-1)^r \varphi(m-1-r; i_1 \neq 1).$$

Proof. By the proof of Proposition 6.21, we have

$$\begin{aligned} & \mathrm{rank}_R E_2^{m-1,0}(\mathrm{B}) \\ = \ & \varphi(m-1; i_1 \neq 1, i_{m-1} \neq m-1) \\ = \ & \sum_{r=0}^{m-3} (-1)^r \varphi(m-1-r; i_1 \neq 1) + (-1)^{m-2} \varphi(1; i_1 \neq 1, i_1 \neq 1) \\ = \ & \sum_{r=0}^{m-2} (-1)^r \varphi(m-1-r; i_1 \neq 1), \end{aligned}$$

which is what we wanted. $\square$

Proposition 6.24. *We have*

$$\mathrm{rank}_R E_2^{m-1,0}(\mathrm{B}) = (-1)^m + \sum_{k=0}^{m-1} (-1)^k (k+1) \varphi(m-1-k).$$

Proof. By Lemma 6.17 and Proposition 6.23, we have

$$\begin{aligned} & \mathrm{rank}_R E_2^{m-1,0}(\mathrm{B}) \\ = \ & \sum_{r=0}^{m-2} (-1)^r \varphi(m-1-r; i_1 \neq 1) \\ = \ & \sum_{r=0}^{m-2} (-1)^r \sum_{s=0}^{m-1-r} (-1)^s \varphi(m-1-r-s) \\ = \ & \sum_{k=0}^{m-1} (-1)^k (k+1) \varphi(m-1-k) - (-1)^{m-1} \varphi(0). \end{aligned}$$

This completes the proof. $\square$

Finally, let us consider the case when $q \geq m$.

Proposition 6.25. *For $q \geq m$, we have*

$$\mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) = \sum_{r=0}^{m-1} (-1)^r \varphi(q-r; i_1 \neq 1).$$

Proof. As in the proof of Proposition 6.21, we have

$$\begin{aligned}
& \mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) \\
= \; & \varphi(q; i_1 \neq 1, i_q \neq m-1) \\
= \; & \sum_{r=0}^{s-1} (-1)^r \varphi(q-r; i_1 \neq 1) + (-1)^s \varphi(q-s; i_1 \neq 1, i_{q-s} \neq m-(s+1)) \\
= \; & \sum_{r=0}^{m-3} (-1)^r \varphi(q-r; i_1 \neq 1) + (-1)^{m-2} \varphi(q-(m-2); i_1 \neq 1, i_{q-(m-2)} \neq 1) \\
= \; & \sum_{r=0}^{m-2} (-1)^r \varphi(q-r; i_1 \neq 1) + (-1)^{m-1} \varphi(q-(m-2); i_1 \neq 1, i_{q-(m-2)} = 1) \\
= \; & \sum_{r=0}^{m-1} (-1)^r \varphi(q-r; i_1 \neq 1),
\end{aligned}$$

which is what we wanted. □

Proposition 6.26. *For $q \geq m$, we have*

$$\mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) = \sum_{k=0}^{m-1} (-1)^k (k+1) \varphi(q-k) + (-1)^m \varphi(q-m+1).$$

Proof. By Lemma 6.17 and Proposition 6.25, we have

$$\begin{aligned}
& \mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) \\
= \; & \sum_{r=0}^{m-1} (-1)^r \varphi(q-r; i_1 \neq 1) \\
= \; & \sum_{r=0}^{m-1} (-1)^r \sum_{s=0}^{m-1} (-1)^s \varphi(q-r-s) \\
= \; & \sum_{k=0}^{2(m-1)} (-1)^k C_k \varphi(q-k),
\end{aligned}$$

where

$$C_k = \begin{cases} k+1 & (0 \leq k \leq m-1), \\ 2(m-1)-k+1 & (m \leq k \leq 2(m-1)). \end{cases}$$

By Lemma 4.16, we have

$$\begin{aligned}
& \sum_{k=m}^{2(m-1)} (-1)^k C_k \varphi(q-k) \\
= \; & (-1)^m \sum_{k=1}^{m-1} (-1)^{k-1} (m-k) \varphi((q-m+1)-k) \\
= \; & (-1)^m \varphi(q-m+1).
\end{aligned}$$

This completes the proof. □

Hence, we have proved Theorem 6.20 by Propositions 6.22, 6.24, and 6.26. Therefore, we have finished the proof of Theorem 6.15.

Remark 6.27. We consider the cochain complex

$$(E_1^{*,-(m-1)+q}(\mathrm{B}), d_1),$$

where

$$E_1^{(m-1)-r,-(m-1)+q}(\mathrm{B}) \cong \mathrm{N}^!_{q-r} \otimes_R \mathrm{Gr}^{(m-1)-r}(\mathrm{B}).$$

Since

$$\mathrm{rank}_R E_1^{(m-1)-r,-(m-1)+q}(\mathrm{B}) = (r+1)\varphi(q-r)$$

for $0 \leq r \leq m-1$, the Euler characteristic of $(E_1^{*,-(m-1)+q}(\mathrm{B}), d_1)$ is given by

$$\chi(E_1^{*,-(m-1)+q}(\mathrm{B})) = (-1)^{m-1} \sum_{r=0}^{m-1} (-1)^r (r+1)\varphi(q-r).$$

For $q \in \mathbb{Z}$, we can directly verify that

$$\begin{aligned} \chi(E_2^{*,-(m-1)+q}(\mathrm{B})) &= \mathrm{rank}_R E_2^{0,-(m-1)+q}(\mathrm{B}) + (-1)^{m-1}\mathrm{rank}_R E_2^{m-1,-(m-1)+q}(\mathrm{B}) \\ &= \chi(E_1^{*,-(m-1)+q}(\mathrm{B})) \end{aligned}$$

by Theorems 6.16 and 6.20.

Summarizing the discussion in §6.3, we have the following theorems by Theorems 5.3 and 6.15.

Theorem 6.28. *Let $m \geq 3$. The Hochschild cohomology $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{B}_m(R))$ is a free R-module for $n \geq 0$. For $n \geq 0$, the rank of $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{B}_m(R))$ is given by*

$$\begin{aligned} &\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{B}_m(R)) \\ = \;& 2\varphi(n) + (-1)^{m-1}(m-1)\varphi(n-m+1) + \sum_{k=1}^{m-2} (-1)^k (k+1)\varphi(n-k). \end{aligned}$$

Theorem 6.29. *Let $m \geq 3$. For $n \geq 0$ and $s \in \mathbb{Z}$, $\mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{B}_m(R))$ is a free R-module. Furthermore,*

$$\mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{B}_m(R)) \cong E_2^{n-s,s}(\mathrm{B}_m(R))$$

as R-modules and

$$\begin{aligned} &\mathrm{rank}_R \mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{B}_m(R)) \\ = \;& \begin{cases} \varphi(s) & (n = s), \\ (-1)^m \varphi(s) + \displaystyle\sum_{k=0}^{m-1} (-1)^k (k+1)\varphi(s+m-k-1) & (n = s+m-1), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

6.4. **Freeness of $E_2^{1,q}(\mathrm{N})$.** We have shown that $E_2^{p,q}(\mathrm{N}) = 0$ unless $p = 0, 1, m-1$ (Remark 6.10) and that $E_2^{m-1,q}(\mathrm{N})$ is a finitely generated free module over R by Corollary 6.9. In this subsection, we show that $E_2^{1,q}(\mathrm{N})$ is a free R-module. We also show that $E_2^{0,0}(\mathrm{N}) \cong R$ and $E_2^{0,q}(\mathrm{N}) = 0$ for $q \neq 0$.

Proposition 6.30. *For $q = 0$, $E_2^{0,0}(\mathrm{N}) \cong R$ and $E_2^{0,0}(\mathrm{N}) \to E_2^{0,0}(\mathrm{B})$ is an isomorphism.*

Proof. Let us consider $d_1^{0,0} : E_1^{0,0}(\mathrm{N}) \cong \mathrm{N}^!_0 \otimes_R \mathrm{Gr}^0(\mathrm{N}) \to E_1^{1,0}(\mathrm{N}) \cong \mathrm{N}^!_1 \otimes_R \mathrm{Gr}^1(\mathrm{N})$. For $cI_m \in \mathrm{N}^!_0 \otimes_R \mathrm{Gr}^0(\mathrm{N}) = RI_m$ with $c \in R$,

$$d_1^{0,0}(cI_m) = \sum_{i=1}^{m-1} (y_i c - c y_i) \otimes E_{i,i+1} = 0.$$

Hence, $E_2^{0,0}(\mathrm{N}) \cong \mathrm{Ker} d_1^{0,0} = RI_m \cong R$. Since $E_2^{0,0}(\mathrm{B}) = RI_m$ by Proposition 6.12, $E_2^{0,0}(\mathrm{N}) \to E_2^{0,0}(\mathrm{B})$ is an isomorphism. □

Proposition 6.31. *For $q \neq 0$, $E_2^{0,q}(\mathrm{N}) = 0$.*

Proof. Obviously, $E_2^{0,q}(\mathrm{N}) \cong \mathrm{Ker} d_1^{0,q} = E_1^{0,q}(\mathrm{N}) = 0$ if $q < 0$. We only need to consider the case that $q > 0$. Let us consider $d_1^{0,q} : E_1^{0,q}(\mathrm{N}) \cong \mathrm{N}^!_q \otimes_R \mathrm{Gr}^0(\mathrm{N}) \to E_1^{1,q}(\mathrm{N}) \cong \mathrm{N}^!_{q+1} \otimes_R \mathrm{Gr}^1(\mathrm{N})$ for $q > 0$. Let $z \in \mathrm{Ker} d_1^{0,q}$. Note that $d_1^{0,q}$ can be regarded as a restriction of $d_1^{0,q} : E_1^{0,q}(\mathrm{B}) \to E_1^{1,q}(\mathrm{B}) \cong E_1^{1,q}(\mathrm{N})$. As in the proof of Proposition 6.12, there exists $g_1, g_2, \ldots, g_{m-1} \in \mathrm{N}^!_{q-1}$ such that

$$\begin{aligned} z = {} & y_1 g_1 \otimes E_{1,1} + \{y_2 g_2 + (-1)^q g_1 y_1\} \otimes E_{2,2} + \cdots \\ & + \{y_{m-1} g_{m-1} + (-1)^q g_{m-2} y_{m-2}\} \otimes E_{m-1,m-1} + (-1)^q g_{m-1} y_{m-1} \otimes E_{m,m}. \end{aligned}$$

Using $z \in E_1^{0,q}(\mathrm{N}) \cong \mathrm{N}^!_q \otimes_R RI_m$, we have

$$y_1 g_1 = y_2 g_2 + (-1)^q g_1 y_1 = \cdots = (-1)^q g_{m-1} y_{m-1}. \tag{6.3}$$

The left hand side and the right hand side of (6.3) are contained in $y_1 \mathrm{N}^!_{q-1}$ and $\mathrm{N}^!_{q-1} y_{m-1}$, respectively. Hence, the both sides of (6.3) are contained in $y_1 \mathrm{N}^!_{q-2} y_{m-1}$, while $y_2 g_2 + (-1)^q g_1 y_1 \in y_2 \mathrm{N}^!_{q-1} + \mathrm{N}^!_{q-1} y_1$. This implies that $y_1 g_1 = \cdots = (-1)^{q+1} g_{m-1} y_{m-1} = 0$ and that $z = 0$. Thus, we have $E_2^{0,q}(\mathrm{N}) = \mathrm{Ker} d_1^{0,q} = 0$ for $q > 0$. □

Theorem 6.32. *For $q \in \mathbb{Z}$, $E_2^{1,q}(\mathrm{N})$ is a finitely generated free module over R.*

Proof. We have an exact sequence of cochain complexes

$$0 \longrightarrow E_1^{*,q}(\mathrm{N}; R) \longrightarrow E_1^{*,q}(\mathrm{B}; R) \longrightarrow E_1^{*,q}(\mathrm{B}/\mathrm{N}; R) \longrightarrow 0.$$

This induces a long exact sequence

$$\cdots \longrightarrow E_2^{*,q}(\mathrm{N}; R) \longrightarrow E_2^{*,q}(\mathrm{B}; R) \longrightarrow E_2^{*,q}(\mathrm{B}/\mathrm{N}; R) \longrightarrow E_2^{*+1,q}(\mathrm{N}; R) \longrightarrow \cdots . \tag{6.4}$$

Let $q = 0$. The map $E_2^{0,0}(\mathrm{N}; R) \to E_2^{0,0}(\mathrm{B}; R)$ is an isomorphism by Proposition 6.30. Since $E_2^{1,0}(\mathrm{B}; R) = 0$ by Proposition 6.7, we obtain that $E_2^{1,0}(\mathrm{N}; R) \cong E_2^{0,0}(\mathrm{B}/\mathrm{N}; R)$, which is isomorphic to the finitely generated free R-module B/N by Theorem 6.4.

Let $q \neq 0$. Since $E_2^{0,q}(\mathrm{N}; R) = 0$ and $E_2^{1,q}(\mathrm{B}; R) = 0$ by Propositions 6.31 and 6.7, we obtain an exact sequence

$$0 \longrightarrow E_2^{0,q}(\mathrm{B}; R) \longrightarrow E_2^{0,q}(\mathrm{B}/\mathrm{N}; R) \longrightarrow E_2^{1,q}(\mathrm{N}; R) \longrightarrow 0. \tag{6.5}$$

By the universal coefficient theorem, we have an exact sequence

$$0 \longrightarrow E_2^{1,q}(\mathrm{N}; \mathbb{Z}) \otimes R \longrightarrow E_2^{1,q}(\mathrm{N}; R) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(E_2^{2,q}(\mathrm{N}; \mathbb{Z}), R) \longrightarrow 0.$$

Since $E_2^{2,q}(\mathrm{N}; \mathbb{Z})$ is a free $\mathbb{Z}$-module by Corollary 6.9, we obtain an isomorphism

$$E_2^{1,q}(\mathrm{N}; \mathbb{Z}) \otimes R \xrightarrow{\cong} E_2^{1,q}(\mathrm{N}; R). \tag{6.6}$$

Let k be a field. Note that $E_2^{0,q}(\mathrm{B}; \mathbb{Z})$ and $E_2^{0,q}(\mathrm{B}/\mathrm{N}; \mathbb{Z})$ are finitely generated free $\mathbb{Z}$-modules by Proposition 6.8 and Theorem 6.4, and hence that $\dim_k E_2^{0,q}(\mathrm{B}; k) = \mathrm{rank}_{\mathbb{Z}} E_2^{0,q}(\mathrm{B}; \mathbb{Z})$ and $\dim_k E_2^{0,q}(\mathrm{B}/\mathrm{N}; k) = \mathrm{rank}_{\mathbb{Z}} E_2^{0,q}(\mathrm{B}/\mathrm{N}; \mathbb{Z})$ by $E_2^{1,q}(\mathrm{B}; \mathbb{Z}) = E_2^{1,q}(\mathrm{B}/\mathrm{N}; \mathbb{Z}) = 0$ (Proposition 6.7 and Theorem 6.4) and the universal coefficient theorem. By (6.5) and (6.6), we obtain

$$\begin{aligned} \dim_k E_2^{1,q}(\mathrm{N}; \mathbb{Z}) \otimes k &= \dim_k E_2^{1,q}(\mathrm{N}; k) \\ &= \dim_k E_2^{0,q}(\mathrm{B}/\mathrm{N}; k) - \dim_k E_2^{0,q}(\mathrm{B}; k) \\ &= \mathrm{rank}_{\mathbb{Z}} E_2^{0,q}(\mathrm{B}/\mathrm{N}; \mathbb{Z}) - \mathrm{rank}_{\mathbb{Z}} E_2^{0,q}(\mathrm{B}; \mathbb{Z}). \end{aligned}$$

This shows that $\dim_k E_2^{1,q}(\mathrm{N};\mathbb{Z})\otimes k$ is independent from the field k. Since each $E_1^{p,q}(\mathrm{N};\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module, $E_2^{1,q}(\mathrm{N};\mathbb{Z})$ is finitely generated $\mathbb{Z}$-module. Thus, we see that $E_2^{1,q}(\mathrm{N};\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module by the fundamental theorem of finitely generated abelian groups. Hence $E_2^{1,q}(\mathrm{N};R)\cong E_2^{1,q}(\mathrm{N};\mathbb{Z})\otimes R$ is a finitely generated free R-module. □

6.5. **The rank of $E_2^{p,q}(\mathrm{N})$.** Note that $\mathrm{rank}_R E_2^{p,q}(\mathrm{N}) = 0$ unless $p = 0, 1$ or $m-1$. In this subsection, we calculate the rank of $E_2^{p,q}(\mathrm{N})$ for $p = 0, 1, m-1$, which is a finitely generated free R-module. As a result, we can determine the R-module structure of $\mathrm{HH}^n(\mathrm{N},\mathrm{N})$.

Theorem 6.33. *We have*

$$\mathrm{rank}_R E_2^{0,q}(\mathrm{N}) = \begin{cases} 1 & (q=0), \\ 0 & (q\neq 0), \end{cases}$$

$$\mathrm{rank}_R E_2^{1,q}(\mathrm{N}) = \begin{cases} m-1 & (q=0), \\ (m-2)\varphi(q) & (q\neq 0), \end{cases}$$

$$\mathrm{rank}_R E_2^{m-1,q}(\mathrm{N}) = (-1)^m\varphi(q) + \sum_{k=0}^{m-1}(-1)^k(k+1)\varphi(q+m-k-1).$$

Proof. When $p=0$, recall that $E_2^{0,0}(\mathrm{N})\cong R$ and $E_2^{0,q}(\mathrm{N})=0$ for $q\neq 0$ by Propositions 6.30 and 6.31. When $p=m-1$, since $E_2^{m-1,q}(\mathrm{N})\cong E_2^{m-1,q}(\mathrm{B})$ by Corollary 6.9, we have $\mathrm{rank}_R E_2^{m-1,q}(\mathrm{N}) = \mathrm{rank}_R E_2^{m-1,q}(\mathrm{B})$, which can be calculated by Theorem 6.15.

We consider the case when $p=1$. Recall the proof of Theorem 6.32. We have an exact sequence

$$0 \longrightarrow E_2^{0,q}(\mathrm{N}) \longrightarrow E_2^{0,q}(\mathrm{B}) \longrightarrow E_2^{0,q}(\mathrm{B}/\mathrm{N}) \longrightarrow E_2^{1,q}(\mathrm{N}) \longrightarrow 0.$$

When $q=0$, we have seen that $E_2^{1,0}(\mathrm{N};R)\cong E_2^{0,0}(\mathrm{B}/\mathrm{N};R)\cong \mathrm{B}/\mathrm{N}$. Hence, we obtain

$$\mathrm{rank}_R E_2^{1,0}(\mathrm{N}) = m-1.$$

When $q\neq 0$, since $E_2^{0,q}(\mathrm{N})=0$ by Proposition 6.31, we have

$$\begin{aligned} \mathrm{rank}_R E_2^{1,q}(\mathrm{N}) &= \mathrm{rank}_R E_2^{0,q}(\mathrm{B}/\mathrm{N}) - \mathrm{rank}_R E_2^{0,q}(\mathrm{B}) \\ &= (m-1)\varphi(q) - \varphi(q) \\ &= (m-2)\varphi(q). \end{aligned}$$

Here we used Theorems 6.4 and 6.15. □

Summarizing the discussions above, we have the following theorems by Proposition 6.1 and Theorem 6.33.

Theorem 6.34. *Let $m\geq 3$. The Hochschild cohomology $\mathrm{HH}^n(\mathrm{N}_m(R),\mathrm{N}_m(R))$ is a free R-module for $n\geq 0$. The rank of $\mathrm{HH}^n(\mathrm{N}_m(R),\mathrm{N}_m(R))$ is given by*

$$\begin{aligned} &\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R),\mathrm{N}_m(R)) \\ &= \begin{cases} 2 & (n=0), \\ 2m-4 & (n=1), \\ \varphi(n)+(m-4)\varphi(n-1)+(-1)^m\varphi(n-m+1)+\displaystyle\sum_{k=2}^{m-1}(-1)^k(k+1)\varphi(n-k) & (n\geq 2). \end{cases} \end{aligned}$$

Theorem 6.35. *Let $m \geq 3$. For $n \geq 0$ and $s \in \mathbb{Z}$, $\mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is a free R-module. Furthermore,*

$$\mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{N}_m(R)) \cong E_2^{n-s,s}(\mathrm{N}_m(R))$$

as R-modules and

$$\begin{aligned} &\mathrm{rank}_R \mathrm{HH}^{n,s}(\mathrm{N}_m(R), \mathrm{N}_m(R)) \\ = &\begin{cases} 1 & (n = 0, s = 0), \\ m-1 & (n = 1, s = 0), \\ (m-2)\varphi(s) & (n = s+1, s \neq 0), \\ (-1)^m \varphi(s) + \displaystyle\sum_{k=0}^{m-1} (-1)^k (k+1)\varphi(s+m-k-1) & (n = s+m-1), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

7. Product structure on $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$

In this section, we describe the product structure on $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ for $m \geq 3$. In §7.1, we deal with the case $m = 3$ explicitly, which is different from the case $m \geq 4$. In §7.2, we deal with the case $m \geq 4$ in general. In any case, there exists an augmentation map $\epsilon : \mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R)) \to R$ as an R-algebra homomorphism such that the Kernel $\overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ of ϵ satisfies

$$\overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R)) \cdot \overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R)) = 0.$$

In particular, we see that $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is an infinitely generated algebra over R.

7.1. The case $m = 3$.

In this subsection, we set $m = 3$ and $\mathrm{N} = \mathrm{N}_3(R)$. Recall $\mathrm{N}^! = R\langle y_1, y_2\rangle / \langle y_1 y_2 \rangle$ in §4.2. We define $c(i,j) \in \mathrm{N}^!$ by

$$c(i,j) = y_2^i y_1^j \in \mathrm{N}^!$$

for $i, j \geq 0$. (Set $c(0,0) = 1$.) Then we can describe the homogeneous part $\mathrm{N}^!_n$ of $\mathrm{N}^!$ of degree n by

$$\mathrm{N}^!_n = R\{c(i,j) \mid i,j \geq 0, i+j = n\}$$

for $n \geq 0$. Note that $\varphi(n) = \mathrm{rank}_R \mathrm{N}^!_n = n+1$.

Let us consider the spectral sequence

$$E_1^{p,q} = \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{Gr}^p(\mathrm{N})) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}, \mathrm{N}).$$

By the discussions in §6, we have $E_2^{p,q} \cong E_\infty^{p,q}$ and

$$E_\infty^{0,q} \cong \begin{cases} R\{c(0,0) \otimes I_3\} & (q = 0), \\ 0 & (q \neq 0), \end{cases}$$

$$E_\infty^{1,q} \cong \begin{cases} R\{c(i, q-i+1) \otimes E_{1,2}, c(q+1, 0) \otimes E_{2,3} \mid 0 \leq i < q\} & (q \geq 1), \\ R\{c(0,1) \otimes E_{1,2}, c(1,0) \otimes E_{2,3}\} & (q = 0), \\ 0 & (q < 0), \end{cases}$$

$$E_\infty^{2,q} \cong \begin{cases} R\{c(i,j) \otimes E_{1,3} \mid i+j = q+2, i > 0, j > 0\} & (q \geq 0), \\ R\{c(0,0) \otimes E_{1,3}\} & (q = -2), \\ 0 & (\text{otherwise}). \end{cases}$$

By direct inspection, we obtain the following lemma.

Lemma 7.1. *The element $c(0,0) \otimes I_3 \in E_\infty^{0,0}$ is a unit of the bigraded algebra $E_\infty^{*,*}$. For any $r, s \geq 1$, the product map $E_\infty^{r,q} \otimes_R E_\infty^{s,q'} \to E_\infty^{r+s,q+q'}$ is a zero map.*

We have

$$F^r\mathrm{HH}^*(\mathrm{N},\mathrm{N}) = \mathrm{Im}(\mathrm{HH}^*(\mathrm{N},F^r\mathrm{N}) \longrightarrow \mathrm{HH}^*(\mathrm{N},\mathrm{N}))$$

and

$$E_\infty^{p,q} \cong F^p\mathrm{HH}^{p+q}(\mathrm{N},\mathrm{N})/F^{p+1}\mathrm{HH}^{p+q}(\mathrm{N},\mathrm{N}).$$

The map

$$\mathrm{HH}^0(\mathrm{N},\mathrm{N}) = F^0\mathrm{HH}^0(\mathrm{N},\mathrm{N}) \longrightarrow F^0\mathrm{HH}^0(\mathrm{N},\mathrm{N})/F^1\mathrm{HH}^0(\mathrm{N},\mathrm{N}) \cong E_\infty^{0,0}$$

gives an augmentation map $\epsilon : \mathrm{HH}^*(\mathrm{N},\mathrm{N}) \to R$. Note that

$$\mathrm{HH}^0(\mathrm{N},\mathrm{N}) = R\{c(0,0)\otimes I_3, c(0,0)\otimes E_{1,3}\}$$

and

$$\begin{aligned}\epsilon(c(0,0)\otimes I_3) &= 1,\\ \epsilon(c(0,0)\otimes E_{1,3}) &= 0.\end{aligned}$$

We can identify $F^1\mathrm{HH}^*(\mathrm{N},\mathrm{N})$ with the kernel of ϵ. By Lemma 7.1, the product map

$$F^1\mathrm{HH}^*(\mathrm{N},\mathrm{N}) \otimes_R F^1\mathrm{HH}^*(\mathrm{N},\mathrm{N}) \longrightarrow F^2\mathrm{HH}^*(\mathrm{N},\mathrm{N})$$

is trivial. Hence we obtain the following theorem.

Theorem 7.2. *There is an augmentation map $\epsilon : \mathrm{HH}^*(\mathrm{N},\mathrm{N}) \to R$ such that $\epsilon(c(0,0)\otimes I_3) = 1$ and $\epsilon(c(0,0)\otimes E_{1,3}) = 0$. Let $\overline{\mathrm{HH}}^*(\mathrm{N},\mathrm{N})$ be the kernel of ϵ. Then we have*

$$\overline{\mathrm{HH}}^*(\mathrm{N},\mathrm{N}) \cdot \overline{\mathrm{HH}}^*(\mathrm{N},\mathrm{N}) = 0.$$

7.2. **The case $m \geq 4$.** Let $\mathrm{N} = \mathrm{N}_m(R)$ for $m \geq 4$. Recall that we have a decomposition

$$C^*(\mathrm{N},\mathrm{N}) = \bigoplus_{s\in\mathbb{Z}} C^{*,s}(\mathrm{N},\mathrm{N})$$

which is compatible with the filtration. We regard

$$C^p(\mathrm{N},\mathrm{N}) = \bigoplus_{s\in\mathbb{Z}} C^{p,s}(\mathrm{N},\mathrm{N})$$

as a $\mathbb{Z}$-graded R-module. Then the triple $(C^*(\mathrm{N},\mathrm{N}), d, \{F^rC^*(\mathrm{N},\mathrm{N})\}_{r\geq 0})$ is a filtered differential graded algebra in the category of $\mathbb{Z}$-graded R-modules. Thus, we obtain a multiplicative spectral sequence

$$E_1^{p,q}(\mathrm{N}) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N},\mathrm{N})$$

in the abelian category of $\mathbb{Z}$-graded R-modules (for details, see §3.1 and §3.3).

Lemma 7.3. *Let $m \geq 4$. If $a \in \mathrm{HH}^{1+q,q}(\mathrm{N},\mathrm{N})$ and $b \in \mathrm{HH}^{1+q',q'}(\mathrm{N},\mathrm{N})$, then $ab = 0$ in $\mathrm{HH}^{2+q+q',q+q'}(\mathrm{N},\mathrm{N})$.*

Proof. By Theorem 6.35, we may assume that $q, q' \geq 0$. Let $x \in E_\infty^{1,q,q}$ and $y \in E_\infty^{1,q',q'}$ be elements which represent a and b, respectively. Since $E_2^{2,q+q',q+q'}(\mathrm{N}) = 0$ for $m \geq 4$ by Theorem 6.33, $E_\infty^{2,q+q',q+q'}(\mathrm{N}) = 0$. Hence $xy = 0$, which implies that ab is represented by an element in $E_\infty^{m-1,q+q'-m+3,q+q'}(\mathrm{N})$. By Lemma 6.3, if $m \geq 4$, then $E_\infty^{m-1,q+q'-m+3,q+q'}(\mathrm{N}) = E_1^{m-1,q+q'-m+3,q+q'}(\mathrm{N}) = 0$. Therefore $ab = 0$. □

Recall $I_m \in C^{0,0}(\mathrm{N},\mathrm{N})$ is a generator of $\mathrm{HH}^{0,0}(\mathrm{N},\mathrm{N})$ (*cf.* Proposition 6.30 and Theorem 6.35). By the decomposition $\mathrm{HH}^*(\mathrm{N},\mathrm{N}) = \bigoplus_{n\geq 0, s\in\mathbb{Z}} \mathrm{HH}^{n,s}(\mathrm{N},\mathrm{N})$, we have an augmentation map $\epsilon : \mathrm{HH}^*(\mathrm{N},\mathrm{N}) \to R$ as an R-algebra homomorphism such that $\epsilon(I_m) = 1$ and $\epsilon(\mathrm{HH}^{n,s}(\mathrm{N},\mathrm{N})) = 0$ for $(n,s) \neq (0,0)$. We can identify $F^1\mathrm{HH}^*(\mathrm{N},\mathrm{N})$ with the kernel of ϵ. Using Lemma 7.3, we see that

$\mathrm{HH}^{n,s}(\mathrm{N},\mathrm{N})\cdot\mathrm{HH}^{n',s'}(\mathrm{N},\mathrm{N})=0$ if $(n,s)\neq(0,0)$ and $(n',s')\neq(0,0)$. Hence, we have the following theorem.

Theorem 7.4. *Let $m\geq 4$. There is an augmentation map $\epsilon:\mathrm{HH}^*(\mathrm{N},\mathrm{N})\to R$ such that $\epsilon(I_m)=1$ for $I_m\in\mathrm{HH}^{0,0}(\mathrm{N},\mathrm{N})$ and $\epsilon(\mathrm{HH}^{m-1,-(m-1)}(\mathrm{N},\mathrm{N}))=0$. Let $\overline{\mathrm{HH}}^*(\mathrm{N},\mathrm{N})$ be the kernel of ϵ. Then we have*

$$\overline{\mathrm{HH}}^*(\mathrm{N},\mathrm{N})\cdot\overline{\mathrm{HH}}^*(\mathrm{N},\mathrm{N})=0.$$

Corollary 7.5. *Let $m\geq 3$. The Hochschild cohomology algebra $\mathrm{HH}^*(\mathrm{N},\mathrm{N})$ is an infinitely generated algebra over R.*

Proof. Suppose that there exists a finite set $G=\{x_i\mid 1\leq i\leq l\}$ of generators of $\mathrm{HH}^*(\mathrm{N},\mathrm{N})$ as an R-algebra. We may assume that x_i is contained in $\mathrm{HH}^{n_i,s_i}(\mathrm{N},\mathrm{N})$ for each i. By Theorems 7.2 and 7.4, $x_ix_j=0$ if $(n_i,s_i)\neq(0,0)$ and $(n_j,s_j)\neq(0,0)$. However, $\mathrm{rank}_R\mathrm{HH}^{s+1,s}(\mathrm{N},\mathrm{N})=(m-2)\varphi(s)>0$ for $s>0$ by Theorem 6.35. This implies that G can not generate $\mathrm{HH}^*(\mathrm{N},\mathrm{N})$, which is a contradiction. Hence, $\mathrm{HH}^*(\mathrm{N},\mathrm{N})$ is an infinitely generated algebra over R. □

Remark 7.6. By [5, Theorem 7.3], if $\Lambda=KQ/I$ is an indecomposable monomial algebra over a field K, then $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is a commutative finitely generated K-algebra of Krull dimension at most one, where $\mathcal{N}$ is the ideal of $\mathrm{HH}^*(\Lambda)$ generated by the homogeneous nilpotent elements. In the $\mathrm{N}_m(K)$ case for $m\geq 3$, $\mathcal{N}=\overline{\mathrm{HH}}^*(\mathrm{N}_m(K),\mathrm{N}_m(K))=\mathrm{Ker}\epsilon$, and $\mathrm{HH}^*(\Lambda)/\mathcal{N}\cong K$ has Krull dimension zero.

8. Gerstenhaber bracket on $\mathrm{HH}^*(\mathrm{N}_m(R),\mathrm{N}_m(R))$

In this section, we describe the Gerstenhaber bracket on $\mathrm{HH}^*(\mathrm{N}_m(R),\mathrm{N}_m(R))$.

8.1. Cocycle representatives.

Set $\mathrm{N}=\mathrm{N}_m(R)$ for $m\geq 3$.

Proposition 8.1. *For $q\geq 0$,*

$$E_2^{1,q}(\mathrm{N},\mathrm{N})=R\{y_{i-1}y_I\otimes E_{i-1,i}-(-1)^q y_Iy_i\otimes E_{i,i+1}\mid 1\leq i\leq m,|I|=q\}$$

as R-subquotients of $E_1^{1,q}(\mathrm{N},\mathrm{N})\cong\mathrm{N}^!_{q+1}\otimes_R\mathrm{Gr}^1(\mathrm{N})$. Here, $y_{i-1}y_I\otimes E_{i-1,i}-(-1)^qy_Iy_i\otimes E_{i,i+1}$ is regarded as $-(-1)^qy_Iy_1\otimes E_{1,2}$ if $i=1$ and $y_{m-1}y_I\otimes E_{m-1,m}$ if $i=m$, respectively.

Proof. Let $\mathrm{M}=\mathrm{M}_m(R)$. We have an exact sequence of cochain complexes

$$0\longrightarrow E_1^{*,q}(\mathrm{N},\mathrm{N})\longrightarrow E_1^{*,q}(\mathrm{N},\mathrm{M})\longrightarrow E_1^{*,q}(\mathrm{N},\mathrm{M}/\mathrm{N})\longrightarrow 0.$$

This induces a long exact sequence

$$\cdots\longrightarrow E_2^{*,q}(\mathrm{N},\mathrm{N})\longrightarrow E_2^{*,q}(\mathrm{N},\mathrm{M})\longrightarrow E_2^{*,q}(\mathrm{N},\mathrm{M}/\mathrm{N})\longrightarrow E_2^{*+1,q}(\mathrm{N},\mathrm{N})\longrightarrow\cdots. \tag{8.1}$$

Since

$$\begin{array}{ccccccccccccc}\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & E_1^{0,q}(\mathrm{N},\mathrm{B}) & \longrightarrow & E_1^{1,q}(\mathrm{N},\mathrm{B}) & \longrightarrow & E_1^{2,q}(\mathrm{N},\mathrm{B}) & \longrightarrow & \cdots\\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & E_1^{0,q}(\mathrm{N},\mathrm{M}) & \longrightarrow & E_1^{1,q}(\mathrm{N},\mathrm{M}) & \longrightarrow & E_1^{2,q}(\mathrm{N},\mathrm{M}) & \longrightarrow & \cdots\end{array}$$

is an isomorphism of cochain complexes, $E_2^{1,q}(\mathrm{N},\mathrm{M})\cong E_2^{1,q}(\mathrm{N},\mathrm{B})=0$ by Proposition 6.7. Then there is an surjection

$$\delta:E_2^{0,q}(\mathrm{N},\mathrm{M}/\mathrm{N})\longrightarrow E_2^{1,q}(\mathrm{N},\mathrm{N}), \tag{8.2}$$

Using $E_2^{0,q}(\mathrm{N},\mathrm{M}/\mathrm{N})\cong\mathrm{N}^!_q\otimes_R((\oplus_{i=1}^mRE_{i,i})/RI_m)$, we obtain a set of generators $\{y_I\otimes E_{ii}\mid 1\leq i\leq m,|I|=q\}$ of $E_2^{0,q}(\mathrm{N},\mathrm{M}/\mathrm{N})$. Since $\delta(y_I\otimes E_{ii})=y_{i-1}y_I\otimes E_{i-1,i}-(-1)^qy_Iy_i\otimes E_{i,i+1}$, we can verify the statement. □

By Proposition 8.1, the R-module $E_2^{1,|I|}(\mathrm{N},\mathrm{N})$ is generated by

$$y_{i-1}y_I \otimes E_{i-1,i} - (-1)^{|I|} y_I y_i \otimes E_{i,i+1}$$

for $1 \le i \le m$. Let $\{I_m^*\} \cup \{E_{ij}^* \mid 1 \le i < j \le m\}$ be the dual basis of $\{I_m\} \cup \{E_{ij} \mid 1 \le i < j \le m\}$ of $\mathrm{N}_m(R)$ over R. For $I = (i_1, \ldots, i_{|I|})$, we set

$$E_I^* = E_{i_1,i_1+1}^* E_{i_2,i_2+1}^* \cdots E_{i_{|I|},i_{|I|+1}}^*.$$

If $|I| = 0$, then set $E_\emptyset^* = 1$.

Lemma 8.2. *In the cochain complex $C^*(\mathrm{N},\mathrm{N})$, the cochain*

$$\sum_{1 \le k < i} E_{k,i}^* E_I^* \otimes E_{k,i} - (-1)^{|I|} \sum_{i < k \le m} E_I^* E_{i,k}^* \otimes E_{i,k}$$

is a cocycle.

Proof. The lemma follows from the following calculations:

$$\begin{aligned} & d(E_{i,j}^* E_I^* \otimes E_{i,j}) \\ = \ & \sum_{k<i} E_{k,i}^* E_{i,j}^* E_I^* \otimes E_{k,j} - \sum_{i<k<j} E_{i,k}^* E_{k,j}^* E_I^* \otimes E_{i,j} + (-1)^{|I|} \sum_{j<k} E_{i,j}^* E_I^* E_{j,k}^* \otimes E_{i,k}, \end{aligned}$$

$$\begin{aligned} & d(E_I^* E_{i,j}^* \otimes E_{i,j}) \\ = \ & \sum_{k<i} E_{k,i}^* E_I^* E_{i,j}^* \otimes E_{k,j} - (-1)^{|I|} \sum_{i<k<j} E_I^* E_{i,k}^* E_{k,j}^* \otimes E_{i,j} + (-1)^{|I|} \sum_{j<k} E_I^* E_{i,j}^* E_{j,k}^* \otimes E_{i,k}. \end{aligned}$$

□

We define

$$a(i,I) \in \mathrm{HH}^{|I|+1,|I|}(\mathrm{N},\mathrm{N})$$

to be the cohomology class represented by the cocycle

$$\sum_{1 \le k < i} E_{k,i}^* E_I^* \otimes E_{k,i} - (-1)^{|I|} \sum_{i < k \le m} E_I^* E_{i,k}^* \otimes E_{i,k}. \tag{8.3}$$

Lemma 8.3. *The cohomology class $a(i,I)$ corresponds to*

$$y_{i-1}y_I \otimes E_{i-1,i} - (-1)^{|I|} y_I y_i \otimes E_{i,i+1}$$

under the isomorphism

$$\mathrm{HH}^{|I|+1,|I|}(\mathrm{N},\mathrm{N}) \cong E_2^{1,|I|}(\mathrm{N},\mathrm{N}).$$

Proof. This follows from the fact that

$$\begin{aligned} & \textstyle\sum_{1 \le k < i} E_{k,i}^* E_I^* \otimes E_{k,i} - (-1)^{|I|} \sum_{i < k \le m} E_I^* E_{i,k}^* \otimes E_{i,k} \\ \equiv \ & E_{i-1,i}^* E_I^* \otimes E_{i-1,i} - (-1)^{|I|} E_I^* E_{i,i+1}^* \otimes E_{i,i+1} \end{aligned}$$

in $C^*(\mathrm{N}, \mathrm{Gr}^1\mathrm{N})$. □

Proposition 6.14 shows that the R-module $E_2^{m-1,|J|-(m-1)}(\mathrm{N},\mathrm{N})$ is generated by

$$y_J \otimes E_{1,m}$$

over R. By the direct calculation, we obtain the following lemma.

Lemma 8.4. *In the cochain complex* $C^*(\mathrm{N},\mathrm{N})$*, the cochain*

$$E_J^* \otimes E_{1,m}$$

is a cocycle. The cohomology class represented by $E_J^* \otimes E_{1,m}$ *corresponds to* $y_J \otimes E_{1,m}$ *under the isomorphism* $\mathrm{HH}^{|J|,|J|-(m-1)}(\mathrm{N},\mathrm{N}) \cong E_2^{m-1,|J|-(m-1)}(\mathrm{N},\mathrm{N})$*.*

We define

$$d(J) \in \mathrm{HH}^{|J|,|J|-(m-1)}(\mathrm{N},\mathrm{N})$$

to be the cohomology class represented by $E_J^* \otimes E_{1,m}$.

8.2. **Construction of an R-basis of $\mathrm{HH}^*(\mathrm{N}_m(R),\mathrm{N}_m(R))$.** In this subsection, we construct an R-basis of $\mathrm{HH}^*(\mathrm{N}_m(R),\mathrm{N}_m(R))$. We set

$$\mathbf{1} = [1 \otimes 1] \in \mathrm{HH}^{0,0}(\mathrm{N},\mathrm{N}).$$

By Theorem 6.35, Propositions 6.14 and 8.1, and Lemmas 8.3 and 8.4, $\mathrm{HH}^*(\mathrm{N},\mathrm{N})$ is generated by

$$\{\mathbf{1}\} \cup \{a(i,I) \mid 1 \le i \le m, |I| \ge 0\} \cup \{d(J) \mid |J| \ge 0\}$$

as R-modules. Since

$$\{y_I \otimes E_{1,m} | \; I = (i_1,\dots,i_q), i_1 \ne 1, i_q \ne m-1\}$$

is an R-basis of $E_2^{m-1,q-(m-1)}(\mathrm{B}) \cong E_2^{m-1,q-(m-1)}(\mathrm{N})$ by Proposition 6.14,

$$\{d(J) \mid J = (j_1,\dots,j_q), j_1 \ne 1, j_q \ne m-1\}$$

is an R-basis of $R\{d(J) \mid |J| = q\}$.

Let us consider $\mathrm{HH}^{q+1,q}(\mathrm{N}_m(R),\mathrm{N}_m(R))$ for $q \ge 0$. Set

$$\mathcal{S}(q) = \{I = (i_1,\dots,i_q) \mid 1 \le i_1,\dots i_q \le m-1, y_I \ne 0\}$$

for $q > 0$ and $\mathcal{S}(0) = \{\emptyset\}$. Note that $\sharp\mathcal{S}(q) = \mathrm{rank}_R \mathrm{N}_m(R)^!_q = \varphi(q)$ for $q \ge 0$. By Proposition 8.1 and Lemma 8.3, $\mathrm{HH}^{q+1,q}(\mathrm{N}_m(R),\mathrm{N}_m(R)) \cong E_2^{1,q}(\mathrm{N},\mathrm{N})$ is generated by $\{a(i,I) \mid 1 \le i \le m, I \in \mathcal{S}(q)\}$ for $q \ge 0$. Recall the long exact sequence (6.4) in Theorem 6.32:

$$\cdots \longrightarrow E_2^{0,q}(\mathrm{N},\mathrm{B}) \xrightarrow{\pi} E_2^{0,q}(\mathrm{N},\mathrm{B}/\mathrm{N}) \xrightarrow{\delta} E_2^{1,q}(\mathrm{N},\mathrm{N}) \longrightarrow \cdots . \tag{8.4}$$

For $1 \le i \le m$, let $b(i,I) = E_I^* \otimes E_{i,i} \in E_2^{0,q}(\mathrm{N},\mathrm{B}/\mathrm{N})$ for $q > 0$ and $b(i,\emptyset) = 1 \otimes E_{i,i} \in E_2^{0,0}(\mathrm{N},\mathrm{B}/\mathrm{N})$ for $q = 0$. We see that $\delta(b(i,I)) = a(i,I)$. Since $\sum_{i=1}^m b(i,I) = E_I^* \otimes I_m = 0 \in E_2^{0,q}(\mathrm{N},\mathrm{B}/\mathrm{N})$,

$$a(1,I) + a(2,I) + \cdots + a(m,I) = 0. \tag{8.5}$$

In Definition 6.11, we have defined

$$z(i,I) = y_i y_I \otimes E_{i,i} + (-1)^{q+1} y_I y_i \otimes E_{i+1,i+1} \in E_2^{0,q+1}(\mathrm{N},\mathrm{B})$$

for $I \in \mathcal{S}(q)$ and $1 \le i \le m-1$ (see also Proposition 6.12). Note that $\pi(z(i,I)) = b(i,(i,I)) + (-1)^{q+1} b(i+1,(I,i))$, where π is the R-homomorphism $\pi : E_2^{0,q}(\mathrm{N},\mathrm{B}) \to E_2^{0,q}(\mathrm{N},\mathrm{B}/\mathrm{N})$ in (8.4). Since $\delta(\pi(z(i,I))) = \delta(b(i,(i,I))) + (-1)^{q+1}\delta(b(i+1,(I,i))) = 0$,

$$a(i,(i,I)) + (-1)^{q+1} a(i+1,(I,i)) = 0 \tag{8.6}$$

for $I \in \mathcal{S}(q)$ and $1 \le i \le m-1$.

Let us construct an R-basis of $\mathrm{HH}^{q+1,q}(\mathrm{N},\mathrm{N}) \cong E_2^{1,q}(\mathrm{N},\mathrm{N})$. Let $q = 0$. By Theorem 6.35, $\mathrm{rank}_R \mathrm{HH}^{1,0}(\mathrm{N},\mathrm{N}) = m-1$. The set $\{a(i,\emptyset) \mid 1 \le i \le m-1\}$ is an R-basis of $\mathrm{HH}^{1,0}(\mathrm{N},\mathrm{N})$, since

$$a(m,\emptyset) = -a(1,\emptyset) - a(2,\emptyset) - \cdots - a(m-1,\emptyset)$$

by (8.5).

Let $q > 0$. By Theorem 6.35, $\mathrm{rank}_R \mathrm{HH}^{q+1,q}(\mathrm{N},\mathrm{N}) = (m-2)\varphi(q)$. Set

$$\mathcal{T}(q) = \{(i, I) \mid 1 \le i \le m, I \in \mathcal{S}(q)\}.$$

Note that $\{a(i, I) \mid (i, I) \in \mathcal{T}(q)\}$ generates $\mathrm{HH}^{q+1,q}(\mathrm{N},\mathrm{N})$ as an R-module.

Definition 8.5. For $q > 0$, set

$$\mathcal{T}(q)_i = \{(i, I) \in \mathcal{T}(q) \mid I = (i, J) \text{ for some } J \in \mathcal{S}(q-1)\}$$

for $1 \le i \le m-1$. We also define

$$\begin{aligned} \mathcal{T}(q)_i^0 &= \{(i,(i,J)) \in \mathcal{T}(q)_i \mid y_J y_i = 0\}, \\ \mathcal{T}(q)_i^1 &= \{(i,(i,J)) \in \mathcal{T}(q)_i \mid y_J y_i \neq 0\}. \end{aligned}$$

Note that $\mathcal{T}(q)_i = \mathcal{T}(q)_i^0 \coprod \mathcal{T}(q)_i^1$ for $q > 0$ and $1 \le i \le m-1$.

Lemma 8.6. *Let $1 \le i \le m-1$. For $(i,(i,J)) \in \mathcal{T}(q)_i^0$, $a(i,(i,J)) = 0$. For $(i,(i,J)) \in \mathcal{T}(q)_i^1$,*

$$a(i,(i,J)) + (-1)^q a(i+1,(J,i)) = 0. \tag{8.7}$$

Proof. By direct calculation and (8.6), we can verify the statement. □

Definition 8.7. For $q > 0$, set

$$\begin{aligned} \mathcal{T}^-(q) = &\left(\bigcup_{i=1}^{m-1} \mathcal{T}(q)_i^0\right) \bigcup \left(\bigcup_{i=1}^{m-2} \left\{(i+1,(J,i)) \mid (i,(i,J)) \in \mathcal{T}(q)_i^1\right\}\right) \bigcup \mathcal{T}(q)_{m-1}^1 \\ &\bigcup \left(\bigcup_{i=1}^{m-2} \{(m,(i,J)) \mid (i,(i,J)) \in \mathcal{T}(q)_i\}\right) \\ &\bigcup \left\{(m,(m-1,J) \mid (m-1,(m-1,J)) \in \mathcal{T}(q)_{m-1}^0\right\} \\ &\bigcup \left\{(1,(m-1,J) \mid (m-1,(m-1,J)) \in \mathcal{T}(q)_{m-1}^1\right\} \end{aligned}$$

and

$$\mathcal{T}(q)^+ = \mathcal{T}(q) \setminus \mathcal{T}(q)^-.$$

Note that $\sharp\mathcal{T}(q) = m\varphi(q)$, $\sharp\mathcal{T}(q)^- = 2\sum_{i=1}^{m-1} \sharp\mathcal{T}(q)_i = 2\varphi(q)$, and $\sharp\mathcal{T}(q)^+ = (m-2)\varphi(q)$ for $q > 0$.

Proposition 8.8. *For $q > 0$, $\{a(i, I) \mid (i, I) \in \mathcal{T}(q)^+\}$ is an R-basis of* $\mathrm{HH}^{q+1,q}(\mathrm{N},\mathrm{N}) \cong E_2^{1,q}(\mathrm{N},\mathrm{N})$.

Proof. Set $T = R\{a(i, I) \mid (i, I) \in \mathcal{T}(q)^+\}$. Let us show that $T = \mathrm{HH}^{q+1,q}(\mathrm{N},\mathrm{N})$. It suffices to prove that $a(i, I) \in T$ for any $(i, I) \in \mathcal{T}(q)^-$. If $(i, I) \in \left(\bigcup_{i=1}^{m-1} \mathcal{T}(q)_i^0\right)$, then $a(i, I) = 0 \in T$ by Lemma 8.6. We easily see that $\mathcal{T}(q)_i^1 \subset \mathcal{T}(q)^+$ for $1 \le i \le m-2$. Put $\mathcal{T}^-(q)' = \bigcup_{i=1}^{m-2} \left\{(i+1,(J,i)) \mid (i,(i,J)) \in \mathcal{T}(q)_i^1\right\}$. If $(i, I) \in \mathcal{T}^-(q)'$, then $a(i, I) \in T$ by (8.7). If $(i, I) \in \bigcup_{i=1}^{m-2} \{(m,(i,J)) \mid (i,(i,J)) \in \mathcal{T}(q)_i\}$, then $a(i, I) \in T$, since

$$R\left\{ a(i,(i,J)) \;\middle|\; (i,(i,J)) \in \bigcup_{i=1}^{m-2} \mathcal{T}(q)_i \right\} + R\{a(i, I) \mid (i, I) \in \mathcal{T}^-(q)'\} \subseteq T$$

and

$$a(i, I) = -\sum_{j \neq i} a(j, I) \tag{8.8}$$

by (8.5). For $(i, I) \in \left\{(m,(m-1,J) \mid (m-1,(m-1,J)) \in \mathcal{T}(q)_{m-1}^0\right\}$, we see that $a(i, I) \in T$ by using $a(m-1,(m-1,J)) = 0$, $a(j,(m-1,J)) \in T$ for $1 \le j \le m-2$ and (8.8). Summarizing the discussion above, we have $a(i, I) \in T$ for any $(i, I) \in \mathcal{T}(q)$ with $i \neq 1, m-1$.

For $(m-1,(m-1,J)) \in \mathcal{T}(q)^1_{m-1}$, let us prove that $a(1,(m-1,J)), a(m-1,(m-1,J)) \in T$. By (8.7) and (8.8),

$$\begin{aligned} a(m-1,(m-1,J)) + (-1)^q a(m,(J,m-1)) &= 0, \\ a(1,(m-1,J)) + a(2,(m-1,J)) + \cdots + a(m-1,(m-1,J)) + a(m,(m-1,J)) &= 0. \end{aligned}$$

Using $a(j,(m-1,J)), a(m,(J,m-1)) \in T$ for $j \neq 1, m-1$, we obtain $a(1,(m-1,J)), a(m-1,(m-1,J)) \in T$.

Hence, $a(i,I) \in T$ for any $(i,I) \in \mathcal{T}(q)^-$ and $T = \mathrm{HH}^{q+1,q}(\mathrm{N},\mathrm{N})$. Since $\mathrm{rank}_R \mathrm{HH}^{q+1,q}(\mathrm{N},\mathrm{N}) = \sharp\mathcal{T}(q)^+ = (m-2)\varphi(q)$, $\{a(i,I) \mid (i,I) \in \mathcal{T}(q)^+\}$ is an R-basis of $\mathrm{HH}^{q+1,q}(\mathrm{N},\mathrm{N})$. $\square$

By the discussion above, we obtain the following corollary.

Corollary 8.9. *We have an R-basis*

$$\begin{aligned} \mathcal{S} \;=\; & \{\mathbf{1}\} \bigcup \{a(1,\emptyset), \ldots, a(m-1,\emptyset)\} \bigcup \left(\bigcup_{q>0} \{a(i,I) \mid (i,I) \in \mathcal{T}(q)^+\} \right) \\ & \bigcup \left(\bigcup_{q \geq 0} \{d(J) \mid J = (j_1, \ldots, j_q), j_1 \neq 1, j_q \neq m-1\} \right) \end{aligned}$$

of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$.

8.3. **Gerstenhaber bracket.** In this subsection, we calculate the Gerstenhaber bracket $[\ ,\ ]$ of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$. By Corollary 8.9, we have the R-basis $\mathcal{S}$ of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$. For investigating the Gerstenhaber bracket on $\mathrm{HH}^*(\mathrm{N},\mathrm{N})$, we only need to calculate $[x,y]$ for $x, y \in \mathcal{S}$.

We easily obtain the following lemmas.

Lemma 8.10. *For any $z \in \mathrm{HH}^{*,*}(\mathrm{N},\mathrm{N})$, we have*

$$[\mathbf{1}, z] = 0.$$

Lemma 8.11. *For any J, J', we have*

$$[d(J), d(J')] = 0.$$

Next, we will calculate $[d(J), a(i,I)]$.

For $I = (i_1, \ldots, i_{|I|})$, $J = (j_1, \ldots, j_{|J|})$, and $1 \leq k \leq |I|$, we set

$$J \underset{k}{\circ} I = (j_1, \ldots, j_{k-1}, i_1, \ldots, i_{|I|}, j_{k+1}, \ldots, j_{|J|}).$$

We also set

$$J(r) = \{k \in \{1, 2, \ldots, |J|\} \mid j_k = r\}.$$

Proposition 8.12. *We have*

$$[d(J), a(i, I)]$$

$$= \begin{cases} (-1)^{|I|} d(I, J) - \sum_{k \in J(1)} (-1)^{k|I|} d(J \underset{k}{\circ} (I, 1)) & (i = 1), \\ \sum_{k \in J(i-1)} (-1)^{(k-1)|I|} d(J \underset{k}{\circ} (i-1, I)) - \sum_{k \in J(i)} (-1)^{k|I|} d(J \underset{k}{\circ} (I, i)) & (1 < i < m), \\ \sum_{k \in J(m-1)} (-1)^{(k-1)|I|} d(J \underset{k}{\circ} (m-1, I)) - (-1)^{|I|(|J|-1)} d(J, I) & (i = m). \end{cases}$$

Next, we will calculate $[a(i, I), a(i', I')]$.

Lemma 8.13. *Let* $x = x^1 + x^2 + \cdots + x^{m-1} \in C^{p+1,p}(\mathrm{N}, \mathrm{N}) \cap \overline{C}^{p+1}(\mathrm{N}, \mathrm{N})$ *and* $y = y^1 + y^2 + \cdots + y^{m-1} \in C^{q+1,q}(\mathrm{N}, \mathrm{N}) \cap \overline{C}^{q+1}(\mathrm{N}, \mathrm{N})$ *be cocycles, where* $x^i \in C^{p+1}(\mathrm{N}, F^i\mathrm{N})$ *and* $y^i \in C^{q+1}(\mathrm{N}, F^i\mathrm{N})$. *Then we have*

$$x \circ y \equiv x^1 \circ y^1 \mod C^{p+q+1}(\mathrm{N}, F^2\mathrm{N}).$$

Proof. Since $x^1 \in C^{p+1,p}(\mathrm{N}, \mathrm{N}) \cap \overline{C}^{p+1}(\mathrm{N}, \mathrm{N})$, x^1 is a linear combination of

$$\{E^*_{i_1,i_1+1} E^*_{i_2,i_2+1} \cdots E^*_{i_{p+1},i_{p+1}+1} \otimes E_{i,i+1} \mid 1 \leq i_1, i_2, \ldots, i_{p+1}, i \leq m-1\}$$

modulo $C^{p+1}(\mathrm{N}, F^2\mathrm{N})$. It is easy to see that $x^1 \circ (y^2 + \cdots + y^{m-1}) \equiv 0 \mod C^{p+q+1}(\mathrm{N}, F^2\mathrm{N})$. Hence, we can verify the statement. □

Lemma 8.14. *Let* $x = x^1 + x^2 + \cdots + x^{m-1} \in C^{p+1,p}(\mathrm{N}, \mathrm{N}) \cap \overline{C}^{p+1}(\mathrm{N}, \mathrm{N})$ *and* $y = y^1 + y^2 + \cdots + y^{m-1} \in C^{q+1,q}(\mathrm{N}, \mathrm{N}) \cap \overline{C}^{q+1}(\mathrm{N}, \mathrm{N})$ *be cocycles, where* $x^i \in C^{p+1}(\mathrm{N}, F^i\mathrm{N})$ *and* $y^i \in C^{q+1}(\mathrm{N}, F^i\mathrm{N})$. *Let* $[x], [y] \in \mathrm{HH}^{*+1,*}(\mathrm{N}, \mathrm{N})$ *be the cohomology classes represented by* x, y, *respectively. Then the element of* $E_2^{1,p+q}(\mathrm{N}, \mathrm{N})$ *that corresponds to the Gerstenhaber bracket* $[[x], [y]] \in \mathrm{HH}^{p+q+1,p+q}(\mathrm{N}, \mathrm{N})$ *is represented by* $x^1 \circ y^1 - (-1)^{(|x|-1)(|y|-1)} y^1 \circ x^1$.

Proof. By Lemma 8.13, $[x, y] \equiv x^1 \circ y^1 - (-1)^{(|x|-1)(|y|-1)} y^1 \circ x^1 \mod C^{p+q+1}(\mathrm{N}, F^2\mathrm{N})$. The proposition follows from the isomorphism $\mathrm{HH}^{p+q+1,p+q}(\mathrm{N}, \mathrm{N}) \cong E_2^{1,p+q}(\mathrm{N}, \mathrm{N})$ since $[[x], [y]] \in \mathrm{HH}^{p+q+1,p+q}(\mathrm{N}, \mathrm{N})$. □

We set

$$A(i, I; i', I')$$

$$= \sum_{k \in I(i'-1)} (-1)^{k|I'|} a(i, I \underset{k}{\circ} (i'-1, I')) - \sum_{k \in I(i')} (-1)^{(k+1)|I'|} a(i, I \underset{k}{\circ} (I', i'))$$
$$-(-1)^{|I||I'|} \left(\sum_{k \in I'(i-1)} (-1)^{k|I|} a(i', I' \underset{k}{\circ} (i-1, I)) - \sum_{k \in I'(i)} (-1)^{(k+1)|I|} a(i', I' \underset{k}{\circ} (I, i)) \right).$$

By direct calculation, we have the following proposition.

Proposition 8.15. *We have*

$$[a(i, I), a(i', I')] = \begin{cases} A(i, I; i', I') & (i \neq i'), \\ A(i, I; i, I') + a(i, (I', I)) - (-1)^{|I||I'|} a(i, (I, I')) & (i = i'). \end{cases}$$

8.4. **Batalin-Vilkovisky structure on** $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$**.** Recall that a Batalin-Vilkovisky algebra is a Gerstenhaber algebra $(A^*, \cup, [\ ,\ ])$ with an operator $\Delta : A^* \to A^{*-1}$ of degree -1 such that $\Delta \circ \Delta = 0$ and

$$[a, b] \quad = \quad (-1)^{|a|}\{\Delta(a \cup b) - \Delta(a) \cup b - (-1)^{|a|} a \cup \Delta(b)\} \tag{8.9}$$

for homogeneous elements $a, b \in A^*$ (see, for example, [1, Definition 3.6]). In this subsection, we consider the question whether the Hochschild cohomology $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ has a Batalin-Vilkovisky algebra structure over R which gives the Gerstenhaber bracket $[\ ,\ ]$ or not.

Lemma 8.16. *Let A be an associated algebra over R such that A is a projective module over R. Assume that $\mathrm{HH}^k(A, A) \cup \mathrm{HH}^l(A, A) = 0$ for any $k, l > 0$. If there exist $a \in \mathrm{HH}^k(A, A)$ and $b \in \mathrm{HH}^l(A, A)$ with $k, l \geq 2$ such that $[a, b] \neq 0$, then $\mathrm{HH}^*(A, A)$ has no Batalin-Vilkovisky algebra structure over R which gives the Gerstenhaber bracket $[\ ,\ ]$.*

Proof. Suppose that $\mathrm{HH}^*(A, A)$ has a Batalin-Vilkovisky algebra structure which gives the Gerstenhaber bracket $[\ ,\ ]$. By (8.9) and $a \cup b = \Delta(a) \cup b = a \cup \Delta(b) = 0$, we obtain $[a, b] = 0$, which is a contradiction. Hence, $\mathrm{HH}^*(A, A)$ has no Batalin-Vilkovisky algebra structure which gives $[\ ,\ ]$. □

Let us show that $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ has no Batalin-Vilkovisky algebra structure over R giving $[\ ,\ ]$ for $m \geq 3$.

Lemma 8.17. *Let $m \geq 3$. For $a(1, (1, 1)), a(1, (2, 1)) \in \mathrm{HH}^{3,2}(\mathrm{N}_m(R), \mathrm{N}_m(R))$,*

$$[a(1, (1, 1)), a(1, (2, 1))] = a(1, (2, 1, 1, 1)) \neq 0.$$

Proof. By Proposition 8.15, $[a(1, (1, 1)), a(1, (2, 1))] = a(1, (2, 1, 1, 1))$. Since $(1, (2, 1, 1, 1)) \in \mathcal{T}(4)^+$, $a(1, (2, 1, 1, 1)) \neq 0$ by Proposition 8.8. □

Theorem 8.18. *For $m \geq 3$, $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ has no Batalin-Vilkovisky algebra structure over R which gives the Gerstenhaber bracket $[\ ,\ ]$.*

Proof. The statement follows from Theorems 7.2 and 7.4, and Lemmas 8.16 and 8.17. □

9. Appendix: the case $m = 2$

In this appendix, we deal with $\mathrm{N}_2(R)$ for a commutative ring R. Set $\mathrm{N} = \mathrm{N}_2(R)$. Putting $x = E_{1,2} \in \mathrm{N}$, we see that $\mathrm{N} \cong R[x]/(x^2)$. Throughout this section, we set $\mathrm{Ann}(2) = \{a \in R \mid 2a = 0\}$. We introduce the following proposition without proof, which gives a projective resolution of N over $\mathrm{N}^e = \mathrm{N} \otimes_R \mathrm{N}^{op}$ over R.

Proposition 9.1 ([4, Proposition 1.3], [13, Example 2.6])**.** *The following complex gives a projective resolution of* N *over* N^e*:*

$$\cdots \longrightarrow \mathrm{N}^e \xrightarrow{d_n} \mathrm{N}^e \longrightarrow \cdots \longrightarrow \mathrm{N}^e \xrightarrow{d_1} \mathrm{N}^e \xrightarrow{\mu} \mathrm{N} \longrightarrow 0, \tag{9.1}$$

where

$$d_i(a \otimes b) = \begin{cases} (x \otimes 1 + 1 \otimes x)(a \otimes b) & (i : \text{even}), \\ (x \otimes 1 - 1 \otimes x)(a \otimes b) & (i : \text{odd}) \end{cases}$$

and $\mu(a \otimes b) = ab$.

In [9], we have calculated $\mathrm{HH}^n(\mathrm{N}_2(R), \mathrm{M}_2(R)/\mathrm{N}_2(R))$ by using the projective resolution above.

Theorem 9.2 ([9, Proposition 4.19])**.** *We have*

$$\mathrm{HH}^n(\mathrm{N}_2(R), \mathrm{M}_2(R)/\mathrm{N}_2(R)) \cong \begin{cases} R \oplus \mathrm{Ann}(2) & (n : \text{even}), \\ R \oplus (R/2R) & (n : \text{odd}). \end{cases}$$

Corollary 9.3 ([9, Corollary 4.20]). *Let k be a field. For each $n \geq 0$,*

$$\mathrm{HH}^n(\mathrm{N}_2(k), \mathrm{M}_2(k)/\mathrm{N}_2(k)) \cong \begin{cases} k & (\mathrm{ch}(k) \neq 2), \\ k^2 & (\mathrm{ch}(k) = 2). \end{cases}$$

By using the same discussions in §5.1 and §5.2, we also have the following result.

Theorem 9.4. *For each $n \geq 0$ and $s \in \mathbb{Z}$,*

$$\mathrm{HH}^{n,s}(\mathrm{N}_2(R), \mathrm{M}_2(R)/\mathrm{N}_2(R)) = \begin{cases} R & (n : \mathrm{even}, s = n), \\ \mathrm{Ann}(2) & (n : \mathrm{even}, s = n+1), \\ R/2R & (n : \mathrm{odd}, s = n), \\ R & (n : \mathrm{odd}, s = n+1), \\ 0 & (\mathrm{otherwise}). \end{cases}$$

Next, let us consider $\mathrm{HH}^n(\mathrm{N}_2(R), \mathrm{N}_2(R))$. By taking $\mathrm{Hom}_{\mathrm{N}^e}(-, \mathrm{N})$ of (9.1), we obtain the following complex

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{N}^e}(\mathrm{N}^e, \mathrm{N}) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathrm{N}^e}(\mathrm{N}^e, \mathrm{N}) \xrightarrow{d_2^*} \mathrm{Hom}_{\mathrm{N}^e}(\mathrm{N}^e, \mathrm{N}) \xrightarrow{d_3^*} \cdots,$$

which is isomorphic to

$$0 \longrightarrow \mathrm{N} \xrightarrow{\delta'^1} \mathrm{N} \xrightarrow{\delta'^2} \mathrm{N} \xrightarrow{\delta'^3} \cdots,$$

where $\delta'^i : \mathrm{N} \to \mathrm{N}$ is defined by

$$\delta'^i(a) = \begin{cases} 2xa & (i : \mathrm{even}), \\ 0 & (i : \mathrm{odd}). \end{cases}$$

Thus, we obtain

Theorem 9.5. *We have*

$$\mathrm{HH}^n(\mathrm{N}_2(R), \mathrm{N}_2(R)) \cong \begin{cases} \mathrm{N}_2(R) & (n = 0), \\ \mathrm{N}_2(R)/(2E_{1,2}\mathrm{N}_2(R)) \cong R \oplus (R/2R) & (n : \mathrm{even}, n > 0), \\ RE_{1,2} \oplus \mathrm{Ann}(2) I_2 \cong R \oplus \mathrm{Ann}(2) & (n : \mathrm{odd}). \end{cases}$$

Notice that $\mathrm{HH}^n(\mathrm{N}_2(R), \mathrm{N}_2(R))$ is not a free R-module in general, which is different from the case $\mathrm{N}_m(R)$ for $m \geq 3$.

Third, let us consider the product structure on $\mathrm{HH}^n(\mathrm{N}_2(R), \mathrm{N}_2(R))$. Set $\overline{\mathrm{N}} = \mathrm{N}/RI_2 \cong Rx$. Recall the reduced bar complex $\overline{B}_p(\mathrm{N}, \mathrm{N}, \mathrm{N}) = \mathrm{N} \otimes_R \overline{\mathrm{N}}^{\otimes p} \otimes_R \mathrm{N}$. Let us consider a homomorphism of chain complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathrm{N} \otimes_R \overline{\mathrm{N}}^{\otimes 2} \otimes_R \mathrm{N} & \longrightarrow & \mathrm{N} \otimes_R \overline{\mathrm{N}} \otimes_R \mathrm{N} & \longrightarrow & \mathrm{N} \otimes_R \mathrm{N} & \xrightarrow{\mu} & \mathrm{N} & \longrightarrow & 0 \\ & & f_2 \downarrow & & f_1 \downarrow & & f_0 \downarrow & & \| & & \\ \cdots & \xrightarrow{d_3} & \mathrm{N}^e & \xrightarrow{d_2} & \mathrm{N}^e & \xrightarrow{d_1} & \mathrm{N}^e & \xrightarrow{\mu} & \mathrm{N} & \longrightarrow & 0, \end{array} \tag{9.2}$$

where $f_p : \overline{B}_p(\mathrm{N}, \mathrm{N}, \mathrm{N}) = \mathrm{N} \otimes_R \overline{\mathrm{N}}^{\otimes p} \otimes_R \mathrm{N} \to \mathrm{N}^e$ is the N^e-homomorphism defined by $f_p(x^{\otimes p}) = I_2 \otimes I_2$ for $p \geq 0$. By taking $\mathrm{Hom}_{\mathrm{N}^e}(-, \mathrm{N})$ of (9.2), we have a quasi-isomorphism of cochain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{N} & \xrightarrow{\delta'^1} & \mathrm{N} & \xrightarrow{\delta'^2} & \mathrm{N} & \xrightarrow{\delta'^3} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{C}^0(\mathrm{N}, \mathrm{N}) & \xrightarrow{\delta^1} & \overline{C}^1(\mathrm{N}, \mathrm{N}) & \xrightarrow{\delta^2} & \overline{C}^2(\mathrm{N}, \mathrm{N}) & \xrightarrow{\delta^3} & \cdots, \end{array}$$

where

$$\overline{C}^p(\mathrm{N}, \mathrm{N}) = \mathrm{Hom}_{\mathrm{N}^e}(\mathrm{N} \otimes_R \overline{\mathrm{N}}^{\otimes p} \otimes_R \mathrm{N}, \mathrm{N}) \cong \mathrm{Hom}_R(\overline{\mathrm{N}}^{\otimes p}, \mathrm{N}).$$

For $p \geq 0$, we define $f_p, g_p \in \overline{C}^p(\mathrm{N}, \mathrm{N})$ by

$$\begin{array}{rcl} f_p(x^{\otimes p}) & = & I_2, \\ g_p(x^{\otimes p}) & = & E_{1,2}, \end{array}$$

respectively. Then $\overline{C}^p(\mathrm{N}, \mathrm{N}) = Rf_p \oplus Rg_p$. By rephrasing Theorem 9.5, we obtain

Theorem 9.6. *For $n \geq 0$, we have*

$$\mathrm{HH}^n(\mathrm{N}_2(R), \mathrm{N}_2(R)) \cong \begin{cases} Rf_0 \oplus Rg_0 & (n = 0), \\ Rf_n \oplus (R/2R)g_n & (n : \text{even}, n > 0), \\ \mathrm{Ann}(2)f_n \oplus Rg_n & (n : \text{odd}). \end{cases}$$

By direct calculation, we obtain the following theorem.

Theorem 9.7. *For any $a \in \mathrm{HH}^*(\mathrm{N}_2(R), \mathrm{N}_2(R))$, $f_0 a = af_0 = a$. For $i, j \geq 0$, we have*

$$\begin{array}{rcccl} f_i f_j & = & f_j f_i & = & f_{i+j}, \\ f_i g_j & = & g_j f_i & = & g_{i+j}, \\ & & g_i g_j & = & 0. \end{array}$$

Remark 9.8. For an odd integer $i > 0$, if $a_i \in \mathrm{Ann}(2)$, then $a_i f_i \in \mathrm{HH}^i(\mathrm{N}_2(R), \mathrm{N}_2(R)) \cong \mathrm{Ann}(2)f_i \oplus Rg_i$. For an even integer $j > 0$, $(a_i f_i)g_j = a_i g_{i+j} \in \mathrm{HH}^{i+j}(\mathrm{N}_2(R), \mathrm{N}_2(R)) \cong \mathrm{Ann}(2)f_{i+1} \oplus Rg_{i+j}$ is well-defined.

Remark 9.9. Theorem 9.7 is compatible with the result in [6, Theorem 7.1]: Let k be a commutative ring with $\mathrm{ch}(k) = p$, where p is a prime number or 0. For $A_2 = k[X]/(X^2)$, the Hochschild cohomology ring of A_2 has the following structure

$$\mathrm{HH}^*(A_2) \cong \begin{cases} k[x, y, z]/(x^2, y^2 - z) & \text{if } p = 2 \\ k[x, y, z]/(x^2, 2xz, yx, y^2) & \text{if } p \neq 2 \text{ and } 2 \in k^\times, \end{cases}$$

where $\deg x = 0, \deg y = 1$, and $\deg z = 2$. (Note that t in [6, Theorem 7.1] is needed to be regarded as n.)

Remark 9.10. We can show that $\mathrm{HH}^*(\mathrm{N}_2(R), \mathrm{N}_2(R))$ is a finitely generated algebra over R if and only if $\mathrm{Ann}(2)$ is a finitely generated ideal of R. Indeed, the "only if" part follows from that $\mathrm{HH}^1(\mathrm{N}_2(R), \mathrm{N}_2(R)) = R \oplus \mathrm{Ann}(2)$. If $\mathrm{Ann}(2) = Ra_1 + \cdots + Ra_s$, then $\mathrm{HH}^*(\mathrm{N}_2(R), \mathrm{N}_2(R))$ is generated by

$$\{f_0, g_0, a_1 f_1, \ldots, a_s f_1, g_1, f_2\}$$

as an R-algebra. In particular, if R is a noetherian ring, then $\mathrm{HH}^*(\mathrm{N}_2(R), \mathrm{N}_2(R))$ is a finitely generated algebra over R.

By calculating $E_2^{p,q,s}(\mathrm{N}_2(R), \mathrm{N}_2(R))$ in §6.1 directly, we also have the following result.

Theorem 9.11. *For each $n \geq 0$ and $s \in \mathbb{Z}$,*

$$\mathrm{HH}^{n,s}(\mathrm{N}_2(R), \mathrm{N}_2(R)) = \begin{cases} R & (n = 0, s = -1), \\ R & (n : \text{even}, n = s \geq 0), \\ R/2R & (n : \text{even}, n = s + 1 \geq 2), \\ \mathrm{Ann}(2) & (n : \text{odd}, n = s \geq 1), \\ R & (n : \text{odd}, n = s + 1 \geq 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Finally, let us consider the Gerstenhaber bracket on $\mathrm{HH}^n(\mathrm{N}_2(R), \mathrm{N}_2(R))$. We can easily verify the following result.

Theorem 9.12. *We have*

$$\begin{aligned}
[f_i, f_j] &= 0 \quad (i, j \geq 0),\\
[f_i, g_j] &= \begin{cases} 0 & (i : \text{even},\ j : \text{even}),\\ f_{i+j-1} & (i : \text{odd},\ j : \text{even}),\\ i f_{i+j-1} & (j : \text{odd}), \end{cases}\\
[g_i, g_j] &= \begin{cases} 0 & (i : \textit{even},\ j : \text{even}),\\ -(j-1) g_{i+j-1} & (i : \text{odd},\ j : \text{even}),\\ (i-1) g_{i+j-1} & (i : \text{even},\ j : \text{odd}),\\ (i-j) g_{i+j-1} & (i : \text{odd},\ j : \text{odd}). \end{cases}
\end{aligned}$$

Suppose that R is a field k of characteristic $\mathrm{ch}(k) \neq 2$. Then

$$\mathrm{HH}^n(\mathrm{N}_2(k), \mathrm{N}_2(k)) \cong \begin{cases} kf_0 \oplus kg_0 & (n = 0),\\ kf_n & (n : \text{even}, n > 0),\\ kg_n & (n : \text{odd}), \end{cases}$$

where f_0 is the unit and $f_{2n} = f_2^n$, $g_{2n+1} = f_2^n g_1 = g_1 f_2^n$, $f_2 g_0 = g_0 f_2 = g_0 g_1 = g_1 g_0 = 0$ for $n \geq 0$. In particular, $\mathrm{HH}^*(\mathrm{N}_2(k), \mathrm{N}_2(k))$ is generated by g_0, g_1, f_2 as a k-algebra.

Theorem 9.13. *Let k be a field of characteristic $\mathrm{ch}(k) \neq 2$. For $c \in k$, define an operator $\Delta_c : \mathrm{HH}^*(\mathrm{N}_2(k), \mathrm{N}_2(k)) \to \mathrm{HH}^{*-1}(\mathrm{N}_2(k), \mathrm{N}_2(k))$ by*

$$\begin{aligned}
\Delta_c(f_0) = \Delta_c(g_0) &= 0,\\
\Delta_c(g_1) &= f_0 + c g_0,\\
\Delta_c(f_{2n}) = \Delta_c(f_2^n) &= 0 \qquad (n > 0),\\
\Delta_c(g_{2n+1}) = \Delta_c(f_2^n g_1) &= (2n+1) f_2^n \qquad (n \geq 0).
\end{aligned}$$

Then Δ_c gives $\mathrm{HH}^*(\mathrm{N}_2(k), \mathrm{N}_2(k))$ a Batalin-Vilkovisky algebra structure which induces $[\ ,\ ]$. In particular, Batalin-Vilkovisky algebra structures on $\mathrm{HH}^*(\mathrm{N}_2(k), \mathrm{N}_2(k))$ giving $[\ ,\ ]$ are not unique.

Proof. By direct calculation, we can verify (8.9). □

Suppose that R is a field k of characteristic $\mathrm{ch}(k) = 2$. Then

$$\mathrm{HH}^n(\mathrm{N}_2(k), \mathrm{N}_2(k)) \cong kf_n \oplus kg_n \qquad (n \geq 0),$$

where f_0 is the unit and $f_n = f_1^n$, $g_n = f_1^n g_0 = g_0 f_1^n$, $g_0^2 = 0$ for $n \geq 0$. In particular, $\mathrm{HH}^*(\mathrm{N}_2(k), \mathrm{N}_2(k))$ is generated by g_0, f_1 as a k-algebra.

Theorem 9.14. *Let k be a field of characteristic $\mathrm{ch}(k) = 2$. For $c, c' \in k$, define an operator $\Delta_{c,c'} : \mathrm{HH}^*(\mathrm{N}_2(k), \mathrm{N}_2(k)) \to \mathrm{HH}^{*-1}(\mathrm{N}_2(k), \mathrm{N}_2(k))$ by*

$$\begin{aligned}
\Delta_{c,c'}(f_{2n}) = \Delta_{c,c'}(g_{2n}) &= 0 \qquad (n \geq 0),\\
\Delta_{c,c'}(f_{2n+1}) &= c f_{2n} + c' g_{2n} \qquad (n \geq 0),\\
\Delta_{c,c'}(g_{2n+1}) &= -f_{2n} + c g_{2n} \qquad (n \geq 0).
\end{aligned}$$

Then $\Delta_{c,c'}$ gives $\mathrm{HH}^*(\mathrm{N}_2(k), \mathrm{N}_2(k))$ a Batalin-Vilkovisky algebra structure which induces $[\ ,\ ]$. In particular, Batalin-Vilkovisky algebra structures on $\mathrm{HH}^*(\mathrm{N}_2(k), \mathrm{N}_2(k))$ giving $[\ ,\ ]$ are not unique.

Proof. By direct calculation, we can verify (8.9). □

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